Modular Arithmetic for Developing Fact Fluency

Tessa Murphy Nitkin

Lewis Elkin Elementary School

Abstract

Modular arithmetic practice allows elementary students to strengthen their ability to unitize, or make groups of numbers, in order to improve their fact fluency with sums between 10 and 20. Students will experience and explore concrete, representational, and abstract forms of modular arithmetic starting with clocks. They will practice and apply these skills through interactive games and group activities.

Keywords

Mathematics, fact fluency, addition, elementary, intervention, modular arithmetic, clock arithmetic

Unit Content

As a math interventionist at Lewis Elkin Elementary School, my curriculum unit is for a small group of 4th grade students who within our Multi-Tiered System of Supports (MTSS) are receiving tier 2 math intervention small group instruction. These groups are working on counting strategies and developing additive mental fact fluency within 20, which is a second grade standard and is necessary for accessing grade level content in the upper grades (Achieve the Core).

More specifically, a lot of my students need to build fact fluency and ultimately automaticity with addition facts with sums in between 10 and 20 that require regrouping. In order to reach automaticity with these facts, students need to first develop conceptual understanding and then fluency regrouping a ten (Van de Walle et al., 2013). While this unit was designed for my older elementary intensive intervention students, the standards, skills built, and prerequisite skills would also be accessible and appropriate for 2nd or 3rd graders in a traditional classroom setting.

In the "What Makes Something a Number?" seminar, we explored modular systems of numbers with a fixed number of results that loop, like a clock does in modulus 12, and the ramifications of various operations on these number systems. This curriculum unit will lead students through modular addition and subtraction conceptually and through practice activities, starting with the engaging and relevant example of clock arithmetic. At the end of the unit, students will connect modular arithmetic to standard arithmetic and explore how working in other systems can help them flexibly unitize. Unitizing is an important part of number sense and is the ability to see groups of objects as a unit which is essential for developing a base ten understanding. The ability to unitize allows students to use early transitional strategies to add and subtract using base ten representations (Ebby et al.).

Teaching Strategies

Throughout the curriculum unit there is a focus on students deriving mathematical concepts as opposed to receiving invented notions. The curriculum will use the strategy of connecting concrete, representational, and abstract forms of modular arithmetic. Students will use concrete examples of modular arithmetic with demonstration clocks, representational examples of cyclical diagrams, and abstract standard modular arithmetic notation.

Some lessons will also incorporate a known, wonder, learn strategy or a notice and wonder strategy in order to engage students and assess prior knowledge. Lessons will also use the think pair share strategy to elicit participation from all students and provide access to the materials by allowing wait time to think.

Based on the high student population of English Language Learners at Lewis Elkin Elementary School, several visuals will be used throughout instruction to increase access. One common visual strategy for all students is use of a ten frame which will be used at the end of the unit to connect modular learning to base ten.

The evaluative tools will include formative and summative assessments. Throughout the unit students will be assessed on their ability to understand modular arithmetic through concrete, representational, and abstract forms. Teachers will also use anecdotal notes from class conversations and games to monitor students' progress. If this unit is being used for a fact fluency intervention group, I would also suggest a pre and post fluency assessment particularly targeting addition facts with sums between 10 and 20.

Classroom Activities

Lesson 1: Introduction to Clock Arithmetic

Time

45 minutes

Objectives

Students will be able to distinguish between traditional counting structures and cyclical modulus 12 clock counting.

Standards

3.MD.A.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2.OA.B.2 Fluently add and subtract within 20 using mental strategies.* By end of Grade 2, know from memory all sums of two one-digit numbers.

Evaluation tool

Appendix C Clock Arithmetic Exploration

Procedures

Know, Wonder, Learn Warm Up

The teacher will start the lesson by presenting an analog clock (with standard Arabic numerals, not Roman numerals) or image of an analog clock and leading students through what they know and wonder about the image while recording their thoughts on an anchor chart. Among other noticing's and background knowledge of time and clocks, students will likely note that the numbers 1-12 are written in a circle and after 12 is 1 again. *See appendix A.*

Clock Arithmetic Exploration

Teacher will ask a few passages of time questions within 12: If Saffiyah gets on the train at 2:00 and it is a 3 hour train ride, what time will she get off the train? If Kevin puts a cake in the oven at 9:00 and it needs 1 hour to bake, what time will it come out of the oven? If Jayden gets to school at 7:00 and stays 4 hours what time will he leave? Students will record these by drawing arrows on an image of an analog clock. See Appendix B. The teacher will model this passing of time with a demonstration clock if available.

Once students are confident within 12, the teacher will review or introduce (based on notice and wonder assessment of students' clock prior knowledge) the concept of AM and PM and the clock recirculating at 12:00 PM and 12:00 AM.

Teacher will ask passage of time questions past 12:00: If Cari gets in the car at 10:00 AM and drives for 4 hours, what time is it now? Pause and ask students: but what is 10+4? Why is 10:00 AM + 4 hours = 2:00 PM? Think pair share.

If Isaiah gets to basketball practice at 11:00 AM and stays for 2 hours, what time does he leave? If Mr. Jones falls asleep at 10:00 PM and sleeps for 8 hours, what time does he wake up? Students will record these by drawing arrows on an image of an analog clock. See Appendix C. The teacher will model this passing of time with a demonstration clock.

Closing

Students will add anything they have learned to them know, wonder, learn anchor chart. This anchor chart will be hung for the rest of the unit for students to refer back to.

Materials

An analog clock or an image of one (Appendix A)

Appendix B-C printed out without one copy for each student

Anchor chart

Demonstration clock if possible

Lesson 2: Continuation of Clock Arithmetic

Time

45 minutes

Objectives

Students will be able to calculate clock arithmetic fluently.

Standards

3.MD.A.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2.OA.B.2 Fluently add and subtract within 20 using mental strategies.* By end of Grade 2, know from memory all sums of two one-digit numbers.

Evaluation tool

Anecdotal notes of students playing the clock arithmetic game.

Procedures

Clock Arithmetic Practice

The teacher will introduce the clock arithmetic game and model how to play. Students will play the clock arithmetic game while building their clock arithmetic fluency. As needed, the teacher will use the demonstration clock to support students developing mod 12 number sense.

Materials

- Clock arithmetic game
- Demonstration clock if possible

*Note: if students are not confident with clock arithmetic by the end of lesson 2, the teacher could spend an extra day playing clock arithmetic games before moving on.

Lesson 3: Other Mods

Time

45 minutes

Objectives

Students will be able to describe what distinguishes modular systems.

Standards

3.MD.A.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2.OA.B.2 Fluently add and subtract within 20 using mental strategies.* By end of Grade 2, know from memory all sums of two one-digit numbers.

Evaluation tool

Modular Arithmetic Representational Practice

Procedures

Notice and Wonder Warm Up

The teacher will present the image of the modulus 8 clock found in Appendix E and ask students to share with a partner what they notice and wonder.

Modular Arithmetic Introduction

Teacher will show a traditional modulus 12 clock. Teacher will ask students: what number do we start over after on this clock? Teacher will label this clock as Mod 12.

Teacher will show a modulus 8 clock and ask students what they would label this clock as. Class will explore together how to add on a modulus 8 clock and the teacher will record the arithmetic in an equation. An alien who uses a modulus 8 clock goes to sleep at 6:00 and wakes up 5 hours later. What time does the alien wake up? *(6 + 5) modulo 8 = 3.*

Modular Identification Game

Students will play a matching game where they pair images of clocks with different modules and they're modulo notation.

Materials

Appendix E: Modulus 8 clock image

Appendix F: Modular Clock Memory Game

Demonstration clock if possible

Lesson 4: Modular Arithmetic Practice- Representational

Time

45 minutes

Objectives

Students will be able to calculate modular addition using representational aids.

Standards

3.MD.A.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2.OA.B.2 Fluently add and subtract within 20 using mental strategies.* By end of Grade 2, know from memory all sums of two one-digit numbers.

Evaluation tool

Modular Arithmetic Representational Practice (Appendix G)

Procedures

Modular Arithmetic Representational Practice

Students will practice modular arithmetic of various mods 8, 5, and 7 using the representational form of a clock diagram.

After students finish, ask them in pairs or whole group to share any trends they noticed. They might notice that there is an identity property in modular arithmetic where for mod n, n + k = k. They also might be able to visualize the mod repeating and not need to notate on the clock model. This is unitizing and is a big step for developing number sense!

Materials

Appendix G: Modular Arithmetic Representational Practice

Lesson 5: Modular Arithmetic Practice- Abstract

Time

45 minutes

Objectives

Students will be able to calculate modular addition using abstract equations.

Standards

3.MD.A.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2.OA.B.2 Fluently add and subtract within 20 using mental strategies.* By end of Grade 2, know from memory all sums of two one-digit numbers.

Evaluation tool

Modular Arithmetic Abstract Practice (Appendix H)

Procedures

Think Pair Share Opening

The teacher will display a mod 6 clock that has arrows representing (3 + 6) mod 5 = 4. Teacher will ask students in pairs to come up with a way to record this arithmetic **without** drawing a clock.

Modular Arithmetic Abstract Introduction and Practice

The teacher will introduce the language of "mod" denoting the total number of a set of integers in a loop, like a clock. Students will practice modular arithmetic of various mods 4, 3, and 8 using the abstract notation of "(a + b) mod c = d." Some students might benefit from the additional support of drawing representations of clocks or using a concrete demonstration clock to calculate the sums.

Materials

Appendix H: Modular Arithmetic Abstract Practice

Lesson 6: Modular Arithmetic Practice Continued

Time

45 minutes

Objectives

Students will be able to flexibly calculate modular addition using representations and abstract equations.

Standards

3.MD.A.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2.OA.B.2 Fluently add and subtract within 20 using mental strategies.* By end of Grade 2, know from memory all sums of two one-digit numbers.

Evaluation tool

Modular Arithmetic Representational Practice

Procedures

Modular Arithmetic Abstract Practice

Students will practice modular arithmetic of various mods 4, 3, 10, and 8 using the abstract notation of "$(a + b) \bmod c = d$." Some students might benefit from the additional support of drawing representations of clocks or using a concrete demonstration clock to calculate the sums.

Materials

Appendix G: Modular Arithmetic Abstract Practice

Lesson 7: Connecting Modular Arithmetic to Base Ten Arithmetic

Time

45 minutes

Objectives

Students will be able to use the unitizing skills they have developed with modular arithmetic in order to fluently add in base ten with sums between 10 and 20.

Standards

3.MD.A.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2.OA.B.2 Fluently add and subtract within 20 using mental strategies.* By end of Grade 2, know from memory all sums of two one-digit numbers.

Evaluation tool

Base ten fact practice.

Procedures

Connecting Representations of Modular and Base Ten Arithmetic

The teacher will revisit the clock anchor chart from Lesson 1. The clock restarts after 12, what do our numbers do after 10? The teacher will show a singular ten frame on the board. Students will count each spot (and the teacher will label with a numeral) as a counter is added and once there are ten realize that another ten frames is needed. The teacher will add another ten frames to the board and another and count with students up until 30. Students will do a think pair share to answer the question "what is similar about these two models? What is different about these two models?" The class will discuss how both models have a set number that requires a "start over" of some type while the clock starts over with the same numerals on the same visual clock and the ten frames start over on a new copy of the ten frames.

Practicing Mod 10 Addition to Base Ten

Students will practice a few mod 10 addition problems before transitioning to parallel problems in base 10. Students will then note what is similar about the solutions and what is different.

Materials

Appendix H: Mod 10 to Base 10 Addition Practice

Optional Lesson 8: Division Extension

Note: this lesson is optional because it will not work for all intended audiences. For students that have some experience with division, such as 4th graders receiving intervention, this lesson can be a great way to connect the differentiated intervention content to the grade level standards of division.

Time

45 minutes

Objectives

Students will be able to connect modular arithmetic to division with remainders.

Standards

3.MD.A.1 Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

2.OA.B.2 Fluently add and subtract within 20 using mental strategies.* By end of Grade 2, know from memory all sums of two one-digit numbers.

4.OA.A.3 Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Evaluation tool

Modular arithmetic to division practice

Procedures

Connecting Modular Arithmetic to Base 10 Division

The teacher will ask students to think-write-pair-share what they wonder and notice about two equations written on the board or projected: 32 mod 6 = 2 and 32 / 6 = 5 remainder 2. Teacher will guide students through a discussion. The teacher might want to include visual models of the problems that students are familiar with. For the modular equation, a mod 6 clock visual could show the 5 loops around the clock with 2 leftover and an equal groups illustration would display the remainder of 2 objects. With both visuals, students can discuss what see similar elements in the images (the leftover 2, the total of 32, the 5 groups or laps, the 6 numbers on the clock or objects in each group) and what different aspects (physical objects versus the passage of time being represented by numbers).

Practicing Translating from Modular Arithmetic to Base 10 Division

Students will independently or in small groups practice the connection between modular number systems and base 10 division problems.

Materials

Appendix I: Mod 10 to Base 10 Division Practice

Resources

Works Cited

Achieve the Core. "Coherence Map." *Coherence Map*, 2022, https://achievethecore.org/coherence-map/2/9.

Ebby, Caroline B., et al. *A Focus on Addition and Subtraction: Bringing Mathematics Education Research to the Classroom*. Routledge, Taylor & Francis Group, 2021.

GB. "Introduction to Clock Arithmetic and Modular Systems." *Math and Multimedia*, 2 Apr. 2016, http://mathandmultimedia.com/2012/07/08/clock-arithmetic/.

Kwong, Harris. "5.7: Modular Arithmetic." *Mathematics LibreTexts*, Libretexts, 7 July 2021, https://math.libretexts.org/Bookshelves/Combinatorics_and_Discrete_Mathematics/A_Spiral_Workbook_for_Discrete_Mathematics_(Kwong)/05%3A_Basic_Number_Theory/5.07%3A_Modular_Arithmetic.

Van De Walle, John, et al. *Elementary and Middle School Mathematics: Teaching Developmentally*. Pearson Education, 2013.

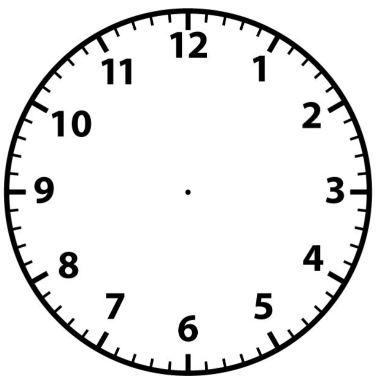
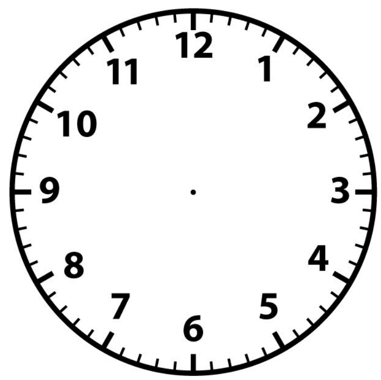
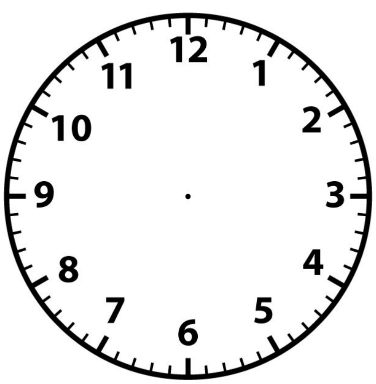
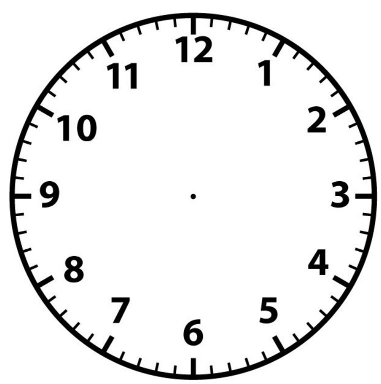
Appendix

Appendix A: Know, Wonder, Learn Analog Clock Anchor Chart

Print out this image to anchor your know, wonder, learn chart.

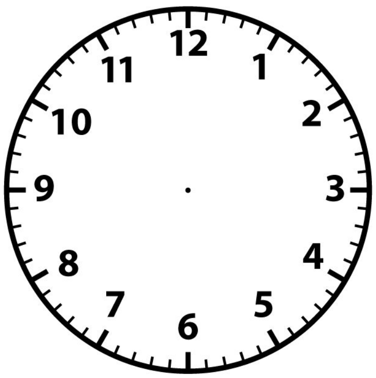
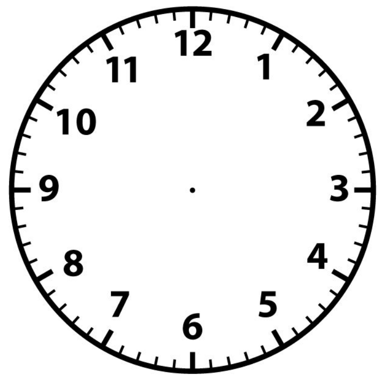
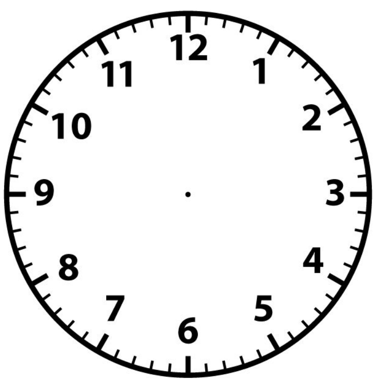
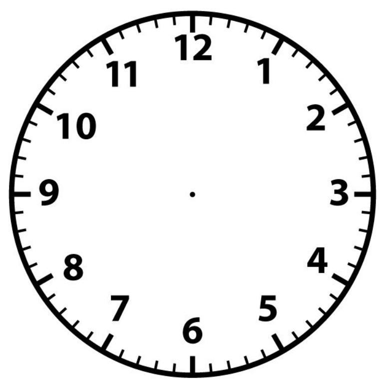
Appendix B: Within 12 passage of time questions

Name: ____

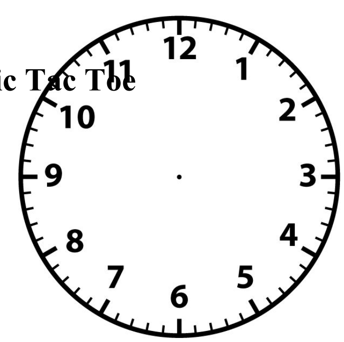
1. If Saffiyah gets on the train at 2:00 and it is a 3 hour train ride, what time will she get off the train?	2. If Kevin puts a cake in the oven at 9:00 and it needs 1 hour to bake, what time will it come out of the oven?
	
3. If Jayden gets to school at 7:00 and stays 4 hours what time will he leave?	4. If Ally put her laundry in the wash at 2:00 and it takes 2 hours, what time will it be done?
	

Appendix C: Beyond 12 passage of time questions

Name: ____

1. If Mr. Jones falls asleep at 10:00 PM and sleeps for 8 hours, what time does he wake up?	2. If Kevin starts playing Pokémon at 11:00 AM and plays for 3 hours, what time will he finish playing?
 ___:___ am / pm	 ___:___ am / pm
3. If the plane leaves at 7:00 AM on a 6 hour flight, what time will it land?	4. If the bread go in the oven at 12:00 PM and needs one hour to bake, what time will it be ready?
 ___:___ am / pm	 ___:___ am / pm

Appendix D: Clock Arithmetic Tic Tac Toe



Materials:

- 1 game board per pair

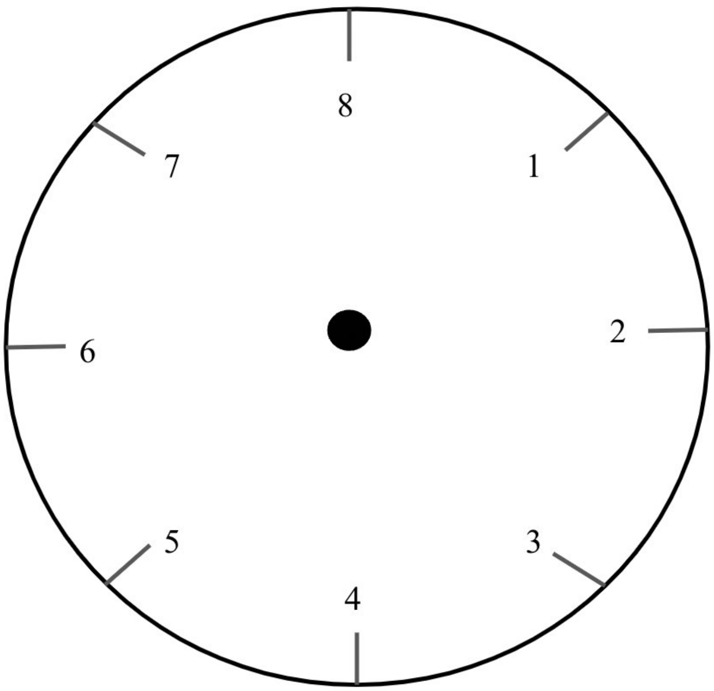
- 2- 9 sided dice per pair of students
- 2 colored counters

Directions: Take turns with your partner. On your turn, roll the two 9-sided dice. Add the two times together in mod 12 on the clock. Cover the answer on the tic tac toe board with your colored counter. Take turns. The first player to 4 in a row (vertically, horizontally, or diagonally) wins! If your sum is already covered, roll again.

1	6	1	6	2
10	2	4	12	5
11	7	3	2	3
7	8	9	4	11
12	10	8	9	5

Note on dice: the game can be played with 6 sided-dice but you may want to use 3 dice at a time to increase the rigor.

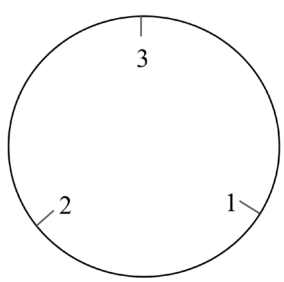
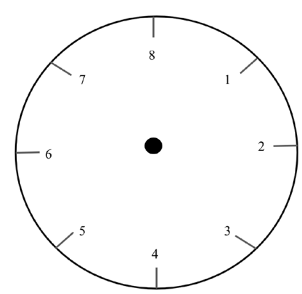
Appendix E: Notice and Wonder Mod 8 Clock

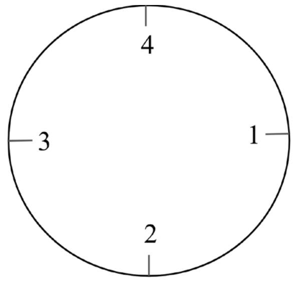
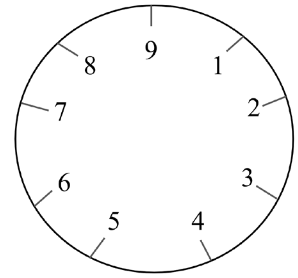
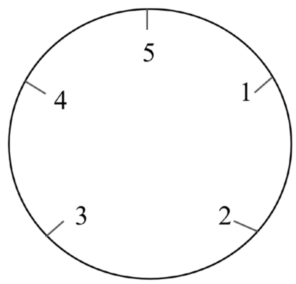
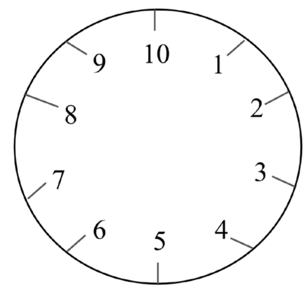
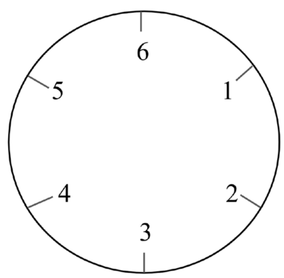
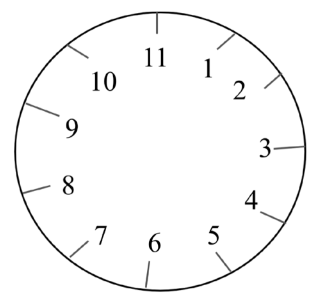
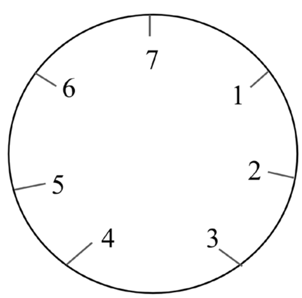
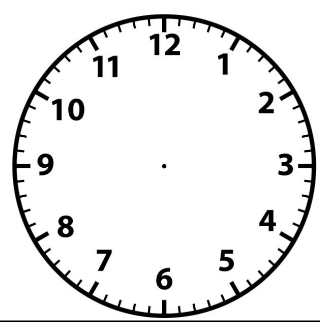


Appendix F: Modular Clock Memory Game

Print and cut out the cards below and shuffle them.

Place the cards face down. On a student's turn they can flip 2 cards. If they match, they can keep them. If not, return them to face down in the same spots. The person with the most cards at the end of the game wins!

Modulus 3 Clock		Modulus 8 Clock	

Modulus 4 Clock		Modulus 9 Clock	
Modulus 5 Clock		Modulus 10 Clock	
Modulus 6 Clock		Modulus 11 Clock	
Modulus 7 Clock		Modulus 12 Clock	

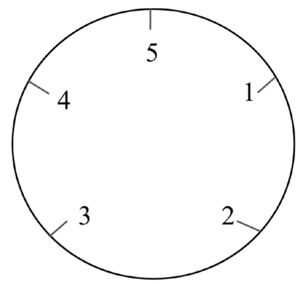
Appendix G: Modular Arithmetic Representational Practice

Name: ____

<table>
<tr><td></td><td></td></tr>
<tr><td>In modulus 5:

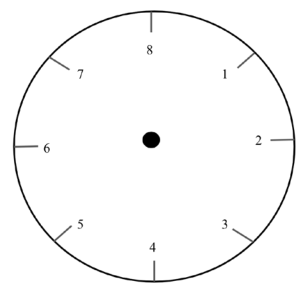
4 + 5 = ___

3 + 3 + 2 = ___

2 + 4 = ___</td><td>
</td></tr>
<tr><td>In modulus 8:

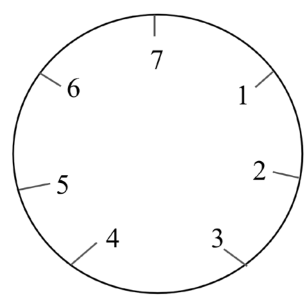
6 + 8 = ___

3 + 7 = ___

1 + 6 = ___</td><td>
</td></tr>
<tr><td>In modulus 7:

2 + 6 = ___

5 + 5 = ___

7 + 3 = ___</td><td>
</td></tr>
</table>

Appendix H: Modular Arithmetic Abstract Practice

Name: ____

1. (5 + 7) mod 8 = ____
2. (3 + 3) mod 4 = ____

3. (2 + 3) mod 3 = ____
4. (1 + 6) mod 8 = ____
5. (2 + 2 + 1) mod 4 = ____
6. (1 + 3) mod 3 = ____

Appendix I: Mod 10 to Base 10 Practice

Name: ____

In modulus 10:	In base 10:
(6 + 5) mod 10 = ___	6 + 5 = ___
(9 + 9) mod 10 = ___	9 + 9 = ___
(2 + 8) mod 10 = ___	2 + 8 = ___
(3 + 9) mod 10 = ___	3 + 9 = ___

Appendix J: Mod 10 to Base 10 Division Practice

Name: ____

In modulus:	In base 10:
43 mod 5 = ___	43 ÷ 5 = ___ remainder __
15 mod 3 = ___	15 ÷ 3 = ___ remainder __

79 mod 10 = ___	79 ÷ 10 = ___ remainder __
25 mod 4 = ___	25 ÷ 4 = ___ remainder __