The Mystery of Dividing by Zero

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Abstract

This curriculum unit asks students what happens when you divide by zero. Students are led through investigation using various definitions of division: division by grouping, division using repeated subtraction, division using inverse of multiplication and finally a limit approach by dividing by incrementally smaller numbers that approach zero. The answer to the main question will be determined by the level of math the student is currently studying. Before students begin their investigations, they will review the properties of adding, subtracting and multiplying by zero with a power-point featuring SuperZero. The unit also includes a link to a Painting with Numbers video and a poem which address what happens when you divide by zero. The final activity is a self-reflection.

Keywords

Division, division by grouping, division by repeated subtraction, operations with zero, undefined, infinity, number representation, math history

Unit Content

Over the years, when I want to get "deep" with a class of students, I will pose the question "Was math invented or discovered?" The answer I receive most often is "math was invented to torture students for 12+ years!" Of course, I was looking for more thought provoked debate. I know that many civilizations dating back thousands of years had estimates for the circumference of a circle divided by its diameter (O'Connor & Robertson, 2001). How can you explain independent exploration into what we now call pi, π? I myself may not have enough information to go too deep, but the push today is for our students to be thinkers, not rote followers of a memorized procedure.

In our TIP class "What Makes Something a Number", led by Dr. Henry Towsner, we go very deep into the inception of a number system and the rules to which a system must adhere. We've examined topics such as modular arithmetic, infinity, imaginary and complex numbers and the real numbers. We discuss how these ideas developed over time and the applications for which they are necessary or utilized. I have learned that many of the limitations we give to students, such as dividing by zero or taking the square root of a negative, really are possible if we change the parameters of a number system in which they can exist!

After the National Council of Teachers of Mathematics recommended a reform in 1989 and the Common Core State Standards for Mathematics were released in 2010 by the National Governors Association Center for Best Practices (NGA Center) and the Council of Chief State School Officers (CCSSO), students' understanding of mathematics became the core of mathematics teaching across North America. The CCSS (Authors, 2010) explains understanding in this way: "One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from." They further emphasize that understanding mathematics is key to success in mastering mathematics. They state: "Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness."

Keeping understanding as goal, my curriculum unit will attempt to deepen high school students' knowledge of the number mysteries of dividing by zero. For instance, instead of accepting that any number divided by zero cannot be answered, students will discover why the quotient is undefined and perhaps explore an instance where division by 0 does exist.

In his article 'Dividing by Nothing' on University of Texas at Austin's website *Not the Past* (Martinez, 2011), Dr. Alberto Martinez discusses the mathematical history of dividing by zero. Three Indian mathematicians in the seventh through twelfth centuries differed in their solutions to dividing by zero. Brahmagupta declared that zero divided by zero is zero. Later, Mahavira included other numbers divided by zero and stated that a number divided by zero is not changed and results in the original number, like another identity element for division. Later still Bhaskara II theorized that a number divided by zero becomes an infinite quantity. In the 17th century, English mathematician John Wallis, who is credited with introducing a sideways 8 as the symbol for infinity, used that symbol as the solution for a number divided by zero. Leonhard Euler, perhaps the most famous of these mathematicians also agreed that a quantity divided by zero is infinity. However, some mathematicians claim that the resulting inverse multiplicative definition results in all numbers being equal. For example, if $5 \div 0 = \infty$ and $6 \div 0 = \infty$, then $0 * \infty$ could be 5 or 6! Clearly, an agreement among scholars is not yet a reality.

In an edition of *The Arithmetic Teacher* from 1961, Marvin Bender reasons in his article "Dividing by Zero" that the answer to a problem where any number other than zero is divided by zero, must be based on the number system the students' use (Bender, 1961). For instance, a fourth grader cannot subtract 10 from 7 and get an answer that is a whole number. If the student has not yet learned to use integers, that problem is impossible or impermissible.

Algebra 1 students cover multiple topics for which dividing by zero could occur. The study of rational functions (when a variable is present in the denominator of a fraction or rational expression) is probably where this division dilemma presents itself the most. In elementary and middle school, students hopefully have learned that dividing by zero is undefined. It follows then when describing the domain of a rational function, it is necessary to restrict this domain to exclude any value for your variable which could result in dividing by zero. In some functions, the resulting graph could have a hole, in other functions, the graph would need asymptotes!

In their article "Teacher Perceptions of Division by Zero" Robert Quinn, Teruni Lamberg and John Perrin (Quinn et al., 2008), found that among the 100 4^{th} – 8^{th} grade teachers they surveyed, 40% could not correctly complete an arithmetic problem where a number other than zero was divided by zero. Another 29% were able to get the right answer, but not explain why. The authors theorized that a teacher's understanding of the concept of dividing by zero is primary before they can help students understand the concept. The need for deeper investigation into this topic is apparent!

Teaching Strategies

Each number mystery investigation could be a stand-alone lesson. The lessons could also be taught consecutively. The intention is to engage in the activities I develop when they come up in the actual topics of the course. I am currently teaching Algebra 1 and Geometry. My intended audience is my Algebra 1 class. Although the timeline to prepare students for the Pennsylvania Keystone Exam, our state's standardized end of course exam required for graduation, in May is quite tight, I believe that taking the time to develop a true understanding of what makes these procedures undefined or not exist will be worth the deeper dive into the topic.

My intent is for the lessons to use discovery learning for students to construct the ideas and properties when investigating about this number mysteries. I plan to include some math history to help spark interest, and to work across disciplines. Students will use class polls and interact with a power point by answering true and false questions.

Students will be provided with a video to enhance the topic development. They will also be introduced to a poem, exploring and summarizing the process of dividing by zero. The lessons conclude with a reflection piece that asks not just what they have learned, but how these investigations about one number mystery have encouraged them to consider other number mysteries, like finding the square root of a negative number!

Classroom Activities

Lesson 1: Dividing by Zero Investigations (1-2 days)

Learning Objective: At the end of this lesson, students will be able to understand by example and investigation, why division by zero is undefined. Depending on the grade level, familiarity with various number systems, and teacher intent, some students will also see how division by zero, using the idea of limits, could approach infinity. Students will review properties of zero.

Materials:

- Power Point: Meet Super Zero (Appendix 1)
- Investigation Worksheet(s)
  - Division by Zero defining Division with Grouping (Appendix 2)
  - Division by Zero defining Division with Repeated Subtraction (Appendix 3)
  - Division by Zero defining Division with Multiplication (Appendix 4)
  - Division by Zero examining Division by Numbers approaching Zero (Appendix 5)

Procedures:

1. Introduce Super Zero by using the power point. Have students review the properties of operating with zero by rating the property on each slide as true or false using hand signals or some other type of answering system.
2. There are 4 investigation sheets that guide students to discover how division by zero is undefined according to the definition or method they use to divide. My suggestion is for Algebra 1 students to complete all 4 in succession. However, the jigsaw strategy could be used to make teams of student experts for each particular method of division.
3. In Appendix 4, a class poll is suggested for different answers to dividing by zero. This is where I will insert some of the math history about dividing by zero. This poll could also be taken earlier in an investigation in appendices 2 or 3. The poll could also be used as a sorting activity to create groups of students that can then discuss their thought process and reasoning.
4. Note: I would not assign these investigations as a homework assignment because the value of communicating and discussing the ideas is very important to developing understanding.

Lesson 2: Extension on Dividing by Zero (1-2 days)

Objectives: At the end of this lesson, students will be able to reflect on their learning on the topic of dividing by zero. Students will also watch a video, illustrating and explaining some of the ideas we covered in our previous investigations. Students will read and reflect on a poem, ‘Dividing by Zero’ by Robin Chapman.

Materials:

- Which One Doesn't Belong (Appendix 6)
- Painting with Numbers, Marcus du Sautoy counts from zero to infinity video link: Zero to Infinity https://www.theguardian.com/science/video/2012/apr/05/marcus-sautoy-counts-zero-infinity-video
- Reflection for Investigation and Poem (Appendix 7)
- Poem: "Dividing by Zero" by Robin Chapman https://poetrywithmathematics.blogspot.com/2011/02/dividing-by-zero.html

Procedures:

1. Introduce the lesson with the Which One Doesn't Belong resource (Appendix 6). The beauty of wodb (wodb.ca) is that there are no wrong answers and multiple entry levels for students. Choices could be simply the shape of the clock or the time represented. When the various selections for different images have been discussed, ask students what the images have in common. The idea is to show different number representations: Roman numerals, Hindu-Arabic numerals, and expressions. This will pave the way to the video's opening discussion of number development.
2. Introduce video, Painting with Numbers, Marcus du Sautoy Counts From Zero to Infinity (*Marcus Du Sautoy Counts from Zero to Infinity - Video | Science | The Guardian*, 2012). While the entire video is worthwhile and relatively short (14:14), the section that covers operating with zero is from 7:14 to 8:35.
3. Distribute the Reflection Questions resource (Appendix 7). Have students answer the 6 reflection questions. This could be assigned for homework if you don't get to it in class.
4. Access and print the poem Dividing by Zero by Robin Chapman (Chapman & Sprott, 2005) on https://poetrywithmathematics.blogspot.com/2011/02/dividing-by-zero.html. Have students read it to themselves twice and then out-loud with their team members. There are 5 reflection questions pertaining to the poem. Encourage students to discuss these with their team before, during or after they answer them.
5. Note: Perhaps you will want students to read and reflect on the poem before they reflect on the investigations.

Resources

Authors. (2010). *Common Core State Standards for Mathematics*. https://learning.ccsso.org/common-core-state-standards-initiative

Bender, M. L. (1961). Dividing by zero Author(s). *Source: The Arithmetic Teacher*, *8*(4), 176–179.

Chapman, R., & Sprott, J. C. (2005). *Images Of A Complex World*. WORLD SCIENTIFIC. https://doi.org/10.1142/5882

Marcus du Sautoy counts from zero to infinity - video | Science | The Guardian. (2012). The Guardian. https://www.theguardian.com/science/video/2012/apr/05/marcus-sautoy-counts-zero-infinity-video

Martinez, A. (2011). *Dividing by Nothing - Not Even Past*. Not Even Past. https://notevenpast.org/dividing-nothing/

O'Connor, J., & Robertson, E. (2001). *Pi history - MacTutor History of Mathematics*. https://mathshistory.st-andrews.ac.uk/HistTopics/Pi_through_the_ages/

Quinn, R. J., Lamberg, T. D., & Perrin, J. R. (2008). Teacher Perceptions of Division by Zero. *The Clearing House: A Journal of Educational Strategies, Issues and Ideas*, *81*(3). https://doi.org/10.3200/tchs.81.3.101-104

Appendix

The Standard for Mathematical Practice (from the Common Core State Standards for Mathematics, CCSS)

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Also, from CCSS

- 3-AO Understand division as an unknown-factor problem.
- F-IF 4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch.
- F-IF 5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

- F-IF 7d (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

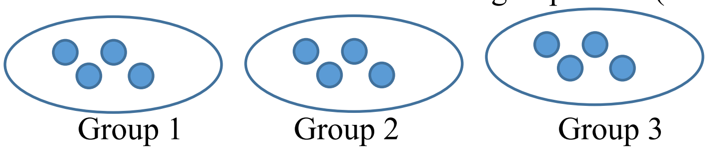
Pennsylvania Academic Standards for Speaking and Listening

CC.1.5.9-10.An Initiate and participate effectively in a range of collaborative discussions on grades level topics, texts, and issues, building on others' ideas and expressing their own clearly and persuasive.

Defining Division by Grouping - 1

Example: Divide 12 into groups of 4, how many groups of 4 will you have?

$12 \div 4 = 3 \because$ 12 can be divided into 3 groups of 4. (this symbol $\because$ means *because*)



Solution: 12 objects can divide into 3 groups of 4.

Complete the following problems and justify division with grouping

1. $8 \div 4 =$ ___ $\because$ 8 can be divided into ____ groups of 4.

Illustration:

2. $15 \div 3 =$ ___ $\because$ 15 can be divided into ____ groups of 3.

Illustration:

3. $10 \div 2 =$ ___ $\because$ 10 can be divided into _____ groups of 2.

Illustration:

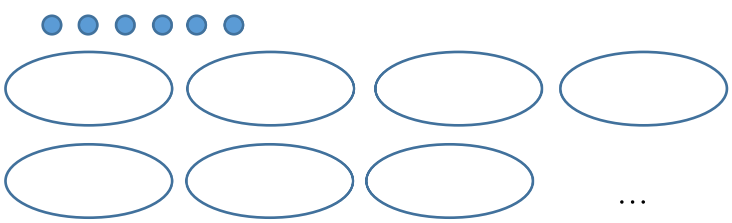
4. $0 \div 5 =$ ___ $\because$ 0 can be divided into ______ groups of 5.

Illustration:

5. $6 \div 0 =$ ___ $\because$ 6 can be divided into ______ groups of 0.

Illustration:

When we use this type of grouping (given number in each group), dividing by zero makes no sense and is <u>undefined</u> for grouping.



Defining Division by Grouping - 2

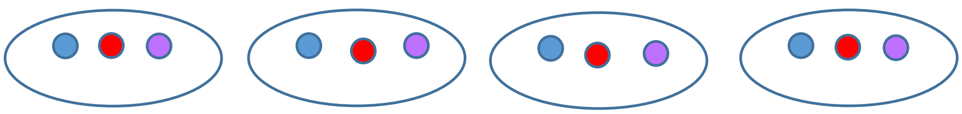
Example: Divide 12 into 4 groups. How much will each group contain?

$12 \div 4 = 3 \because$ 12 when divided into 4 groups will contain 3 each.

(this symbol $\because$ means *because*)

Illustration:

Group 1 Group 2 Group 3 Group 4



Solution: 12 can be divided into 4 groups of 3.

1. $9 \div 3 =$ ___ $\because$ 9 can be divided into 3 groups of _____.
Illustration:

2. $8 \div 4 =$ ___ $\because$ 8 can be divided into 4 groups of _____.
Illustration:

3. $12 \div 2 =$ ___ $\because$ 12 can be divided into 2 groups of ___.
Illustration:

4. $0 \div 3 =$ ___ $\because$ 0 can be divided into 3 groups of _____.
Illustration:

5. $5 \div 0 =$ ___ $\because$ 5 can be divided into 0 groups of _____.
Illustration:

 How can we split 5 into zero groups – that means there would be no groups. Where could we put the 5 objects?

This problem has no explanation or definition when using grouping with a set number of groups and is therefore <u>undefined</u>.

Defining Division with Repeated Subtraction

Example:

counter

$$\begin{array}{rcrcrl} 12 & - & 4 & = & 8 & \quad \mathbf{1} \\ 8 & - & 4 & = & 4 & \quad \mathbf{2} \\ 4 & - & 4 & = & 0 & \quad \mathbf{3} \end{array}$$

$12 \div 4 = 3 \because 12 - 3(4) = 0.$ (this symbol $\because$ means *because*)

We could subtract the divisor 4 three times from the dividend 12.

Complete the following problems and justify with repeated subtraction.

1. $30 \div 6 =$ ____ $\because 30 - 6(\ \ ) = 0$
2. $18 \div 2 =$ ____ $\because 18 - 2$ __ times gives us 0
3. $0 \div 5 =$ ____ $\because 0 - 5(\ \ \ ) = 0$ How many 5's can we subtract from 0 to get 0?
4. $4 \div 0 =$ ____ How many 0's can we subtract from 4 to get 0? Will we ever get 0 by subtracting 0's?
   When we use repeated subtraction to define division, it doesn't matter how many times you subtract 0 from a number (other than 0), you will never reach 0. For this reason, we say dividing by zero is <u>undefined</u>.

Defining Division with Multiplication

We use multiplication to define division. (this symbol ∵ means *because*)

Example: $\frac{12}{4} = 3 \quad \because \quad 4 * 3 = 12$

Complete the following problem and justify the division with multiplication

1. $\frac{20}{2} = $ ___ $\because$ $2 *$ __ $= 20$
2. $\frac{35}{7} = $ __ $\because$ $7 *$ __ $= 35$
3. $\frac{0}{5} = $ ____ $\because$ $5 *$ ____ $= 0$
4. $\frac{6}{0} = $ _____ $\because$ $0 *$ _____ $= 6$
5. What happened with #4? Let's take a class poll…
    a. How many people think the answer is 0?
    b. How many people think the answer is 6?
    c. How many people think the answer is 1?

When we use multiplication to define division, we can't find a Real number, that when multiplied by 0, will give you 6! Therefore, we say that the answer is <u>undefined</u>, because it doesn't fit in the definition of division.

Dividing by Numbers Close to Zero

Calculate these division answers.

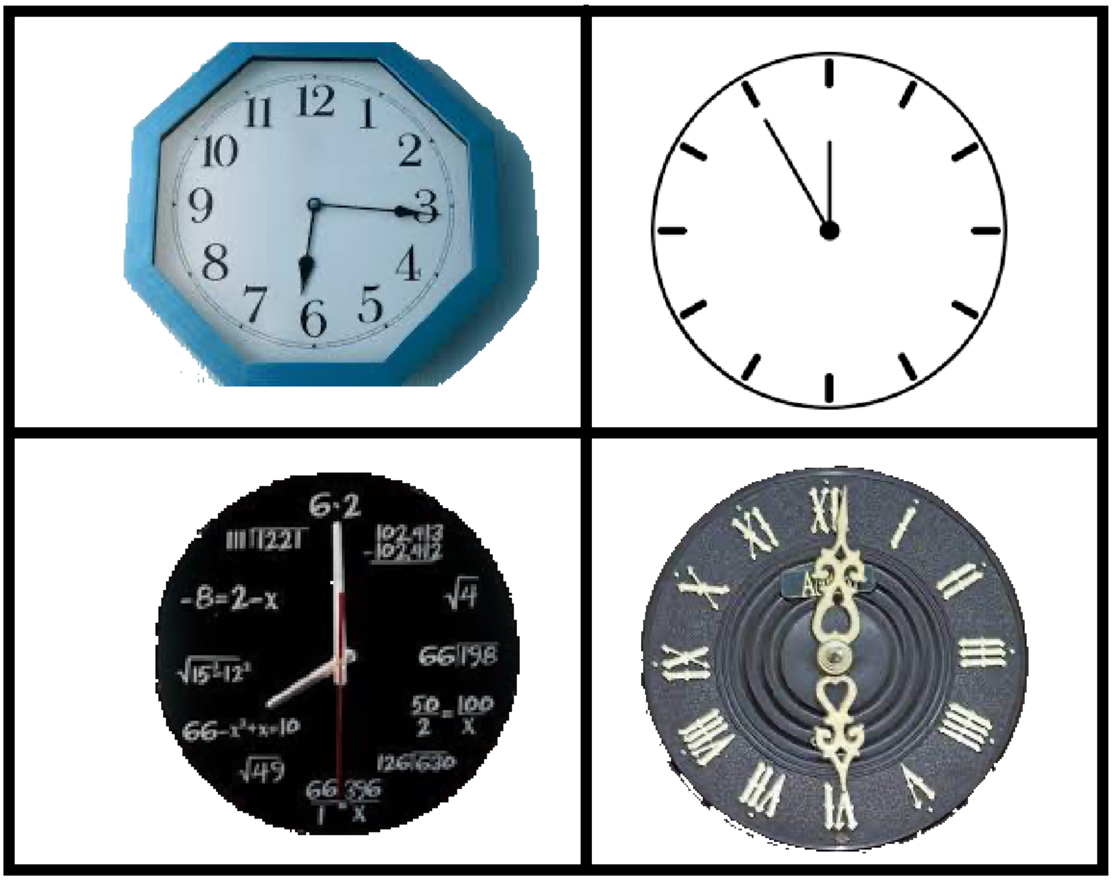
1. $\frac{1}{1} =$ _____
2. $\frac{1}{\frac{1}{2}} =$ ____
3. $\frac{1}{\frac{1}{10}} =$ ____
4. $\frac{1}{\frac{1}{100}} =$ ____
5. $\frac{1}{\frac{1}{1000}} =$ ____

What is happening to our answers as our divisor gets smaller; getting closer and closer to zero?

In higher math, we call this approach **the limit**.

For this reason, we can say that any number (other than 0) divided by zero is infinity. This can only happen when you work in a number system that contains infinity.

Which One Doesn't Belong



The Mystery of Dividing by Zero

Reflection Questions

For this investigation you have:

- Reviewed the rules of operating with zero.
- Learned about the history of dividing by zero.
- Investigated various ways to describe dividing by zero for different explanations and definitions of division and in different number systems.
- Watched a video or video clip about operating with zero and counting from zero to infinity.
- Read aloud, discussed and reflected upon the poem 'Dividing by Zero' by Robin Chapman (you will do this after your initial reflection).

Answer the questions with as much detail as you can.

1. Before this investigation, what did you think about operating with zero?
2. What thoughts or questions did you have prior to this investigation about various number mysteries including dividing by zero? (If you never thought about this or related topics, please explain why.)
3. Why do you think it is important to dive deeply into understanding mathematical dilemmas?
4. Which method or definition of dividing by zero made the most sense to you or did you like the most?
5. What questions do you still have about dividing by zero?
6. What other number mysteries are of interest to you? If you can't think of any, name another mathematical topic you would like to dig deeper into.

Poem Reflection

Read the poem, ‘Dividing by Zero’ twice and then out-loud with your team. Feel free to discuss the questions below as you answer them.

1. Have you ever read a math poem before? If so, what was the topic?
2. Which verse of the poem did you connect to? If you didn’t connect to any of the verses, which verse sounds best to you or uses imagery you like?
3. How does the poem align with our investigations?
4. Interpret the poem’s ending. What is it saying about our mathematical mystery?

It may be surprising that art, music and poetry are intertwined with math. Give one or two examples of math used in other liberal arts subjects?