Redefining the Teaching & Learning of Mathematics

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Abstract

Our goal as math teachers is to strive to achieve high mathematics achievement and equitable outcomes. This research-based curriculum unit proposes that Philadelphia math classrooms need to change the math experiences for our students. So many students come in with fixed mindsets about themselves as math students and about math class due to years of closed math experiences. Many students and adults also have a concept that the goal of math is to get the correct answer. In reality, the core of mathematics is reasoning. The standards for mathematical practices are the guide for this unit, proposing open-ended tasks that encourage student thought, productive struggle, perseverance, collaborating, communication, and the ability to ask questions and defend one's reasoning. By building these skill sets into the math classroom we are creating a more equitable learning environment where all students have an access point, but all students also have an ability to be challenged. We are also creating spaces for students to work authentically and to be active members in their learning. Until these changes happen, we will continue to see math anxiety, fear of math, and a separation of math class from all other subjects.

Keywords: Math Education, Equity, mathematical practices, productive struggle, active learning, problem solving,

Content Objectives

Problem Statement

Ninth grade students typically enter high school with a fear of failure. Students even more specifically have a math failure. For so many students, math class has looked a certain way for the majority of their schooling. The teacher is at the center and the experience is one that can be characterized by a passive means of learning. Students come into high school with a fixed mindset that they are either a math person or not a math person because this is what has been instilled in them via their learned experiences. What if the goal of teaching math was not to have students memorize content or complete long lists of practice problems, but rather a subject that made students think deeply while searching for answers to thought provoking questions? What if instead math class were a space where students are active members of their education, the students are thinking critically, working as a community, building communication skills, learning to collaborate, learning to problem solve, and learning to use math as a tool to read, write, see, and reform the world that they live in.

But, how do we do develop mathematical mindsets in our students that are so afraid of failure? Jo Boaler (2016) writes that mathematics classrooms in the United States are often set up to make students feel good about themselves, by giving them lots of questions that they can answer. Boaler (2016) goes on to write that teachers believe that mistakes and struggle are unproductive and try to shelter them. Mistakes are how our brain grows.

"Research has recently shown when students make a mistake in math, their brain grows, synapses fire, and connections are made; when they do the work correctly, there is no brain growth" (Moser et al. 2011) The brain is a muscle and when we want our muscles to get stronger we have to work them out. If we want to work out our biceps to get stronger, we might do curls, if we want our quads to get stronger, we might do squats, or leg presses. Regardless of the exercise, if you chose a muscle in your body that you wanted stronger you would have to spend time and energy exercising. This is not different for our brains and math. If we want our brains to build math strength and to get stronger, we have to put in the time and exercise. In our classrooms we have to give our students the time to exercise their brains. You have to put in the time to build those habits, processes, and strategies. Building this endurance can be challenging. As a teacher we need to first change the way that students see themselves as math students, and that math class is the space to do the work of a mathematician, critically thinking, problem solving, looking for patterns, asking questions, and at times struggling.

Not only do we need to address the mindsets of the students in the classrooms, but we need to address the structures of the classroom. Like Boaler (2016) addressed, our American math classrooms are not providing learning opportunities for students to productively struggle and critically think and reason. Which means, our classrooms are not providing learning opportunities to build math endurance, to build math strength, and to allow the brain to grow new neural pathways. Why is math class different from other classes? "In order to learn to be a good English student, to read and understand novels, or poetry, students need to have memorized the meanings of many words. But no English student would say or think that learning about English is about the fast memorization and fast recall of words. This is because we learn words by using them in many different situations – talking, reading, and writing" (Boaler, 2015) Students should have the same learning opportunities in the math classroom. Boaler (2015) goes on to write that the core of mathematics is reasoning - thinking through why methods make sense and talking about reasons for the use of different methods. Students need to be given tasks that require questioning, discussion, struggle, collaboration and analysis. Working on these tasks leads students to develop their understanding of concepts and be an active participant in their learning. The teacher serves as the facilitator and support system.

Students have a fear of engaging in tasks that require struggle, students find it difficult to persist through tasks that require multiple iterations, and students view mistakes as a negative. Knowledge gained through research and in the seminar, Democracy and Expertise in Science, History, & Literature will demonstrate the role of failure and struggle in learning, the importance of a growth mindset in mathematics education, and the role that epistemic truth plays in changing the way that students view themselves as a learner.

There are teachers that have tried to incorporate thinking into their math classrooms, while still meeting the district standards and requirements of standardized testing. This comes with much struggle for a teacher. At the center of the is struggle rests beliefs about the nature of mathematics as a discipline and why it is treated differently, questions about mathematics role in our democratic society, and what are the most effective instructional strategies that best support student learning. There are also some big questions remain that unanswered for many:

•Why do so many students encounter failure in mathematics?
• What mathematics do all students really need as citizens of the 21st century?
• How can teachers support students not only in knowing more mathematics but also in having a greater connection to the subject?

Rationale

I work at the Creative & Performing Arts High School (CAPA) in Philadelphia. My students come in with so many preconceived notions about themselves as math students. They see themselves as creative individuals, readers, writers and performers. Seldom do my students see themselves as mathematicians. The missing piece is that everyone is a math person. Everyone can be good at math. Everyone can be successful at math. Yet my students do not often see themselves as math people. My students come in describing all of their terrible experiences with math and all of the reasons why they probably will not be successful. This unit seeks to understand why so many students enter high school with a fear of math class and to change the way that my students view themselves as learners.

This unit is designed for the students at CAPA, which is a performing arts magnet school in the School District of Philadelphia. Because the magnet school falls into the “special admit” category in the school district, students apply, audition, and travel from all over the city, which results in a very diverse student body with regard to ethnic, cultural, and academic backgrounds. At CAPA 59% of the students are black, 21% white, 9% latinx, 6% biracial, and 5% Asian.

I will be able to utilize the newly rewritten vision of the teaching of mathematics at the School District of Philadelphia, as a foundation for this unit.

> "All students think mathematically and they will be empowered to own, share and do mathematics. Our students will have opportunities to learn and participate in meaningful mathematics daily...We provide all students with the opportunity to think and reason mathematically with mathematical concepts that are not explicitly taught but derived and reinforced through student learning experiences." (Math Department, School District of Philadelphia)

The School District of Philadelphia in its vision for math classes specifically notes that our students need to be participating in mathematics and that students need to be given opportunities to critically think, and reason. This unit will expand on that vision and bring the vision to life. Although this unit is designed for the students at CAPA in Algebra 1 and Algebra 2 it can be used and is applicable for any math classroom in the School District of Philadelphia. All math classrooms in Philadelphia need to be structured around the core of math being about reasoning. This unit will explicitly model and allow for the development of the habits and processes of mathematicians.

Standards

The Standards for Mathematical Practice are the foundation for the content standards. These standards create a description of how all students should engage with math as they develop and grow as learners, ensure that all students have a true understanding of math, and encourage all students to develop the reasoning and communication skills required not only in math, but in our everyday lives.

Common Core State Standards (CCSS) for Mathematical Practice (SMP):

Overarching Habits of Mind of a Productive Math Thinker	
SMP #1. Make sense of problems and persevere in solving them	SMP #6. Attend to Precision
Reasoning and Explaining	
SMP #2. Reason abstractly and quantitatively	SMP #3. Construct viable arguments and critique the reasoning of others
Modeling and Using Tools	
SMP #4. Model with mathematics	SMP #5. Use appropriate tools strategically
Seeing Structure and Generalizing	
SMP #7. Look for and make use of structure	SMP #8. Look for and express regularity in repeated reasoning

Background Content for the Unit

Mathematical Practices

The eight standards for mathematical practice (SMP) are the backbone of every common core standard. These standards, and more specifically the habits and practices of a mathematician are the focus of this unit. These are the eight mathematical practices identified under the common core mathematical standards:

SMP1: Make Sense of Problems & Persevere in Solving Them
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They monitor and evaluate their progress and change course if necessary. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

SMP2: Reason Abstractly & Quantitatively
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved.

SMP3: Construct Viable Arguments & Critique the Reasoning of Others
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

SMP4: Model with Mathematics
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They routinely interpret their mathematical results in the context of the situation

and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

SMP5: Use Appropriate Tools Strategically
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

SMP6: Attend to Precision
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. Students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

SMP7: Look for & Make Use of Structure
Mathematically proficient students look closely to discern a pattern or structure. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects

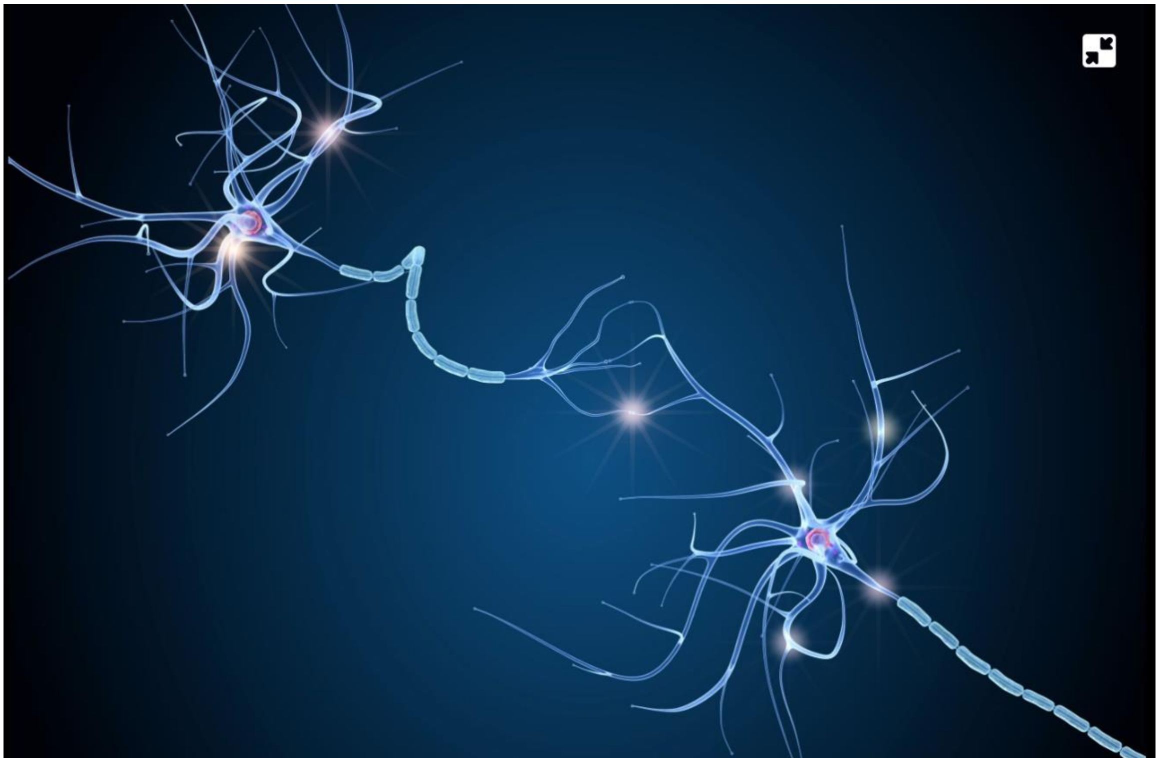
SMP8: Look for & Express Regularity in Repeated Reasoning
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

The Brain & Mathematics Learning
Brains can grow, adapt, and change. This is known as brain plasticity. Woollett & Maguire (2011) have published research on the brain demonstrating the brain's flexibility and ability to change and grow. Woollett & Maguire's (2011) research has shown us that the brain has the ability to grow and change in a really short period of time. When we learn a new idea, an electric current fires in our brain, crossing synapses and connecting different areas of the brain. The research goes on to show that if you learn something deeply, the synaptic activity will create lasting connections in your brain. This

information is important, because translated to the classroom, translates to every student can be successful in math. Jo Boaler (2016) writes that not everyone is born with the same brain, but these brain differences that we are born with are nowhere near as important as the brain growth experiences that we have throughout life. Boaler (2016) goes on to write that there is no such thing as a “math brain” or a “math gift;” no one is born knowing math, and no one is born lacking the ability to learn math. Every single student that comes into our classrooms has the capacity to learn math and the capacity to learn math to high levels. The research on brains demonstrates that a student’s brain is flexible and has the ability to grow and develop new neural connections.

Rishi Sriram (2020) gives an overview of how the learning process happens in the brain in their article, “The Neuroscience Behind Productive Struggle.”

> Learning occurs when experiences connect neurons together. This learning leads to thoughts and behaviors when brain signals travel from neuron to neuron. Imagine a student’s brain as a field. Creating learning through new connections among neurons is like clearing a path and making a dirt road in that field. At first, brain signals can travel, but not quickly or efficiently. But repeated practice—guided by feedback to correct errors—tells the brain that this dirt path is not sufficient. The brain responds by paving the road so that signals can travel faster. Specialized cells called glial cells are the brain’s construction workers. Myelin is the brain’s pavement. When brain signals repeatedly travel through neurons, glial cells create myelin, a fatty substance that wraps around the wires (called axons) connecting the neurons. Myelin plays the essential role of making brain signals faster and stronger. The more the brain signal is “practiced,” the more myelin gets wrapped around the wire. A well-myelinated brain signal travels over 100 times faster than an unmyelinated brain signal. (Sriram, para 5)



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Myelin—a fatty substance that resembles sausage links—wraps around the axon of a neuron, increasing the efficiency of brain signals.

It is then the teacher's job to help the students strengthen their neural pathways. This can happen in the classroom by introducing tasks that require some productive struggle and problem solving. This means introducing tasks in the classroom that require perseverance and effort, reasoning, and sense making. These tasks that require struggle and encourage mistake making are helping the brain to grow and develop its neural pathways. When a student is struggling and perseveres, they may do so my references prior knowledge, or making a connection to something a classmate stated, or by partnering with a peer, or by using a visual a approach, or by trying a different representation. This all leads to pathways that are getting strengthened by myelin production and brain growth.

Mindsets

There are two different types of mindsets, a fixed mindset and a growth mindset. When people change their ideas about the malleability of their potential, from "fixed" (my ability is not changeable) to "growth" (my ability changes as I learn) their learning and achievement improves (Dweck, 2006) Carol Dweck (2006) summarized her research evidence from many years of research with different aged subjects and was able to show that when students develop what she has called a ‘growth mindset’ then they believe that intelligence and ‘smartness’ can be learned and that the brain can grow from exercise. The implications of this mindset are profound -- students with a growth mindset work and learn more effectively, displaying a desire for challenge and resilience in the face of failure. On the other hand, those with a ‘fixed mindset’ believe that you are either smart or you are not. When students with a fixed mindset fail or make a mistake they believe that they are just not smart and give up. Such students frequently avoid challenges, preferring instead to complete easier work on which they know they will succeed.

Boaler (2016) writes that the awareness that ability is malleable and that students need to develop productive growth mindsets has profound implications for teaching. The prevalence of fixed mindset beliefs among students has led to students wanting opportunities to produce pages of correct mathematics work in classrooms. But, as I explain to teachers, if students are producing pages of correct work then their brains are not growing and opportunities for development are missed. Students need to be working on challenging work that results in mistakes; their mistakes should be valued for the opportunities they provide for brain development and learning.

With students we need to support those that come in with a fixed mindset and help them to develop a growth mindset. This can be done through changing how they see themselves as a math student, and the idea that some people are just “math people.” This can also be done through working with students to value their mistakes. Another way to work with a student's mindset is to change the student's language. When a student tells you that they “give up”, ask them to instead say, “I’ll use some of the strategies that I have learned. If a student tells you that something is “too hard” ask them to instead say, “this may take some time and effort.” If a student tells you something is “good enough” ask them to instead say, “I can improve and make this better.” The dialogue that teachers have with teachers can change the mindset that students have about themselves and that can change how students see themselves as learners. As the teacher it is important to be conscious of the language that we use with teachers as well. Praise within the classroom can contribute to a student’s fixed or growth mindset. It is important that teachers are praising the process, that teachers are acknowledging the effort, strategies, and ideas of students. A teacher that just celebrates a correct answer is sending the wrong message to students.

Why is Math Class treated differently?

In order to learn to be a good English student, to read and understand novels, or poetry, students need to have memorized the meanings of many words. But no English student would say or think that learning about English is about the fast memorization and fast recall of words. This is because we learn words by using them in many different situations – talking, reading, and writing. English teachers do not give students hundreds of words to memorize and then test them under timed conditions. All subjects require the memorization of some facts, but mathematics is the only subject in which teachers believe they should be tested under timed conditions. Why do we treat mathematics in this way? One way that mathematics is different is that it is often thought of as a performance subject. Most students think their role in math classrooms is to get questions right. Students rarely think that they are in math class to appreciate the beauty of mathematics, to ask deep questions, to explore the rich set of connections that make up the subject, or even to learn about the applicability of the subject. Students often describe math as a subject of calculations, procedures, or rules. When you ask a mathematician what math is, they will say it is the study of patterns, that it is an aesthetic, creative, and beautiful subject. We have to question why there is such a large discrepancy between these definitions, because when students of English literature are asked what the subject is, they do not give descriptions that are all that different from what professors of English literature would say.

The start of timed testing is the beginning of math anxiety for many. Many people within society have trauma surrounding math or have had bad experiences in a math classroom. But, Jo Boaler (2016) writes that the "negative ideas that prevail about math do not come only from harmful teaching practices. They come from one idea, which is very strong, permeates many societies, and is at the root of math failure and underachievement: that only some people can be good at math. That single belief - that math is a "gift" that some people have and others don't - is responsible for much of the widespread math failure in the world." Those that feel that they do not have this math gift view math people as smarter than themselves and see themselves as a failure over and over again. Many of these students go on to develop anxiety and fears around doing math because they see others as smarter and more capable than themselves because they do not see themselves as smart. They see math people as smart.

Math anxiety for many comes from the fact that people see math differently from other subjects. Many people think that math is a subject of right and wrong answers, a subject based around facts and formulas. Schools have a misguided emphasis on memorization and testing and there is a connection between the math crisis that we currently face. Math facts by themselves are only a small part of mathematics. Unfortunately, many classrooms focus on math facts in isolation. In doing this they miss the beauty, the creativity, and interpretive moments of math. Math is filled with connections.

Through brain research via MRI imaging it has been discovered that math facts are held in the working memory section of the brain. When students are stressed, such as they might be when taking a timed test, the working memory becomes blocked. (Beilock, 2011) When students realize that they cannot perform well on these time tests, they start to develop math anxiety and start to develop negative feelings towards math. Jo Boaler (2016) writes that math anxiety has been recorded in students as young as five, and timed tests are a major cause of this debilitating, often lifelong condition. In her classes at Stanford University, she encounters many undergraduates who have been math traumatized, even though these students are among the highest achieving students in the country. When Boaler asks these students, what led to their math aversions, many talk about timed tests in second or third grade as a major turning point when they decided math was not for them.

The Importance of Mistakes, Failure, & Struggle

There are three elements that are critical to math classrooms. Mistake making, failure, and productive struggle allow for brain growth, conceptual understanding, and deeper learning. Every time that a student makes a mistake in math their brain grows a synapse. Mistakes are useful and they show us that we are learning. Psychologist Jason Moser studied the neural mechanisms that operate in people's brains when they make mistakes. What was most significant from this research was that Moser (2011) discovered that an individual does not have to be aware that they have made a mistake for the brain to spark. Jo Boaler (2016) elaborated on Moser's research and noted that the brain sparks and grows when we make a mistake, even if we are not aware of it, because it is a time of struggle, the brain is challenged, and this is the time when the brain grows the most. This research is important because so many students feel that mistakes are a bad thing. Honestly, even many teachers and adults will view mistakes as a negative. The conclusion that we can draw though, is that making a mistake is a good thing. When you make a mistake your brain grows. Teachers need to think about the work that they are giving to students, they need to design tasks that are open ended and allow for struggle. Too often the work that is given to students in math class is designed so students will get the work correct. This does not leave much room for brain growth.

It is important to also change the way that students view mistakes. Most students view mistakes as a negative. As the classroom teacher we need to teach students that mistakes are positive. It is important to publicly value mistakes in the classroom. Boaler (2016) writes that a teacher can share with a class their favorite mistakes on an assignment and discuss where the mistakes come from and why it is a mistake. Boaler (2016) also suggests talking to students about how their mistakes are a good thing because it means that they were in a state of cognitive struggle and their brain was sparking and growing.

Not only is it important that we change the way that students view mistakes, but we also need to change the way that students view failure. Aislinn O'Donnell (2014) reflects on the ideas and philosophies of Samuel Beckett. O'Donnell (2014, p 262) writes that Beckett's philosophy invites another way of thinking about failure, not as something one is, but rather as something one does. Based on this idea teachers need to support students to separate themselves from their failures. Students need to see themselves as individuals separate from their failures. O'Donnell (2014, p 262) wonders what impact does emphasis on competition and achievement have in contemporary education and its implications for our relationship to failure. Do our education systems place emphasis on the wrong items if students are competing for the highest marks on an exam, or to be the top ranked student in a class? In this system, the emphasis is on scores, not necessarily on the mastery of a topic or the processes that a student uses to approach problems or tasks. Based on what O'Donnell (2014) writes, systems like this have competition rooted in comparison and contribute to relating failure that of one's being and one's failure to match up to others. Students compare themselves based on how they perform, and if they haven't performed as well as their classmate. If a student has failed an assessment and their peer has not, a student is likely to view themselves as a failure. O'Donnell (2014) notes that this relationship to failure raises a range of questions in respect to both students' self-conceptualization and their relation to their peers, to those in authority and to societal norms. If the dominant way of understanding failure in education is to judge success in relation to a standard or norm, then failure is a relative term contingent upon what that standard or norm prescribes. Why is our standard of norm performance based? Why are we not valuing the processes, collaboration, team work, communication, reasoning, and other skills necessary for success in society?

O'Donnell (2014) explores ways of reimagining our relationship to failure in such a way that allows us to reflect on what matters in life. Is getting a certain score on an exam important in the scheme of life? Does it matter if a student scores a 95% on an exam on systems of equations, or does it matter if a student is able to determine when a small business owner will be able to make a profit on their jewelry business of bracelets and necklaces at the craft fair, because the student could be this small business owner.

O'Donnell (2014) also suggests that rather than seeking to eliminate failure, it is more interesting to think about how the concept of failure could be reimagined if, as Beckett suggests, failure is not accompanied by disappointment but becomes simply what we do, rather than being something we are. If we are closed to the possibility of failure, then we are closed to life's openness and unpredictability. We are thus invited to think of failure as an inevitable dimension of being human in the world to which our response is simply to persist. This ability to persevere is what is written into the first standard for mathematical practice of the common core standards. The Standard for Mathematical Practice states that students have the ability to monitor their progress and change course if necessary. If a student is on a task they are able to manage their time and energy toward a goal while monitoring their work. If that student gets to a dead end, they are able to reassess the task, are able to reexplain to themselves the meaning of a problem and look for different entry points to a solution. This raises the question of whether teaching practices and institutions of education need to be reimagined in order to redefine success, so that failure might come to be understood not as being a failure and as a deficit of being a relatively new concept according to Scott Sandage (2010)—but in terms of the activity of failing. Schools cultivate, in his view, producers, rather than thinkers. Producers look for the right answer, certainty and approval and become concerned if they do not receive affirmation or if they get the wrong answer. The structures that exist in many of our classrooms, particularly math classrooms, are performance based. The emphasis needs to shift, so that failure is seen as part of the learning process, and is part of the work of a student, just like it is for mathematicians. Failure is not a negative and it does not mean that you yourself are a failure. It means that a problem or task has not yet been solved and needs a different approach.

Failure is not the only classroom occurrence that needs to be redefined. Many teachers struggle to incorporate productive struggle into their classrooms. There are many different reasons why productive struggle is often missing from the classroom. It could be that students are resistant to struggle and often shut down, but it could also be because teachers too have a difficult time watching students struggle and themselves step in before a student has adequately had time to work through a problem on their own. Many teachers have a hard time shifting the focus to the students and letting the students take the lead. “Teachers sometimes perceive student frustration or lack of immediate success as indicators that they have somehow failed their students. As a result, they jump in to “rescue” students by breaking down the task and guiding students step by step through the difficulties. Although well intentioned, such “rescuing” undermines the efforts of students, lowers the cognitive demand of the task, and deprives students of opportunities to engage fully in making sense of the mathematics” (Reinhart 2000; Stein et al. 2009). When we think about productive struggle it most generally connects with the first common core standard for mathematical practice, “make sense of problems and persevere in solving them.”

According to NCTM’s Principles to Action (NCTM, 2014) the productive struggle in learning mathematics is defined as: Effective teaching of mathematics that consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships. NCTM considers this to be one of the eight essential Mathematics Teaching Practices that provide a framework for strengthening the teaching and learning of mathematics. NCTM (2014) establishes a series of norms about what productive struggle should look like in their publication Principles to Action. NCTM has established expectations for students, practices to support teachers as they facilitate productive struggle, and classroom based indicators of what success looks like.

Expectations for students	Teacher actions to support students	Classroom-based indicators of success
Most tasks that promote reasoning and problem solving take time to solve, and frustration may occur, but perseverance in the face of initial difficulty is important.	Use tasks that promote reasoning and problem solving; explicitly encourage students to persevere; find ways to support students without removing all the challenges in a task.	Students are engaged in the tasks and do not give up. The teacher supports students when they are "stuck" but does so in a way that keeps the thinking and reasoning at a high level.
Correct solutions are important, but so is being able to explain and discuss how one thought about and solved particular tasks.	Ask students to explain and justify how they solved a task. Value the quality of the explanation as much as the final solution.	Students explain how they solved a task and provide mathematical justifications for their reasoning.
Everyone has a responsibility and an obligation to make sense of mathematics by asking questions of peers and the teacher when he or she does not understand.	Give students the opportunity to discuss and determine the validity and appropriateness of strategies and solutions.	Students question and critique the reasoning of their peers and reflect on their own understanding.
Diagrams, sketches, and hands-on materials are important tools to use in making sense of tasks.	Give students access to tools that will support their thinking processes.	Students are able to use tools to solve tasks that they cannot solve without them.
Communicating about one's thinking during a task makes it possible for others to help that person make progress on the task.	Ask students to explain their thinking and pose questions that are based on students' reasoning, rather than on the way that the teacher is thinking about the task.	Students explain their thinking about a task to their peers and the teacher. The teacher asks probing questions based on the students' thinking.

(NCTM, 2014)

Productive struggle actually impacts the brains of students. As discussed above, challenging tasks spur the production of myelin, a substance that increases the strength of brain signals. Productive Struggle leads to an increased production of myelin, which strengthens the connections between neurons. Productive struggle requires students to put in more time and effort, which is enhancing students' learning of concepts.

The best time for our brains is when we are struggling and making mistakes. Neuroscientist Paul Moser found that when people make mistakes their brains are on fire with activity. NCTM's Principles to Action states "Effective mathematics teaching supports students in struggling productively as they learn mathematics. Such instruction embraces a view of students' struggles as opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions". These are the times that the students will be able to make deep connections and build understanding of concepts. Students may be able to make connections to prior knowledge, students may build the skill set of learning to try multiple methods. In these moments of struggle students can build curiosity, a willingness to learn from others, learn to communicate & collaborate, and many other important skills. It is also important to encourage the idea to students that it is okay that they do not currently know something and that together we will work to find it out. This opens up opportunities to create a community of learners with the teacher and student seen both as learners.

In comparisons of mathematics teaching in the United States and in high-achieving countries, U.S. mathematics instruction has been characterized as rarely asking students to think and reason with or about mathematical ideas (Banilower et al. 2006; Hiebert and Stigler 2004). Teachers sometimes perceive student frustration or lack of immediate success as indicators that they have somehow failed their students. As a result, they jump in to "rescue" students by breaking down the task and guiding students step by step through the difficulties. Although well intentioned, such "rescuing" undermines the efforts of students, lowers the cognitive demand of the task, and deprives students of opportunities to engage fully in making sense of the mathematics (Reinhart 2000; Stein et al.2009)

What is Math really?

Mathematics is a conceptual domain. The core of mathematics is reasoning – thinking through why methods make sense and talking about reasons for the use of different methods (Boaler, 2013). It is not, as many people think, a list of facts and methods to be remembered. Developing number sense comes about through a mathematical mindset that is focused on making sense of numbers and quantities.

Reasoning, justification, and problem solving are all critical acts in mathematics learning. Jo Boaler (2016) writes that she has met many teachers who tell her they would love to highlight these core aspects of mathematics more but feel they cannot because of the pressures of content coverage, pacing guides, and district tests. Why there is still so much focus on standardized tests and the pressure to cover extensive lists of content in districts, when this is inconsistent with what the research says about how students should be learning math makes one question what the priority is.

Math is a sense-making subject. Students have the ability to know if an answer is right by reasoning about it. Math is about being creative. If a student finishes a question, there is always the opportunity to think of a different way to solve it or to come up with another question.

Wolfram (2010) the director of Wolfram-Alpha writes that working on mathematics has four stages: 1) posing a question, 2) going from the real world to a mathematical model, 3) performing a calculation, 4) going from the model back to the real world, to see if the original question was answered. When students are given 30 problems to work on from a textbook there is not opportunity to work through these four stages, students typically have the opportunity to only perform stage 3. Even when students see a word problem from their textbook it typically only requires stage 3. Most of these stages are often absent from the classrooms.

Math is a tool to be able to read, write, and see the world. Students need to be learning how to use this tool in classrooms. In his article, "Democracy & School Math," Kurt Stemhagen (2011), references the philosophies of John Dewey:

> "Dewey's conceptualization of mathematics is humanistic and pragmatic; that is, he saw mathematics as a set of tools human have constructed to solve real problems in an ongoing effort to live better. This way of thinking about mathematics, affords students the opportunity to engage in genuine problem solving and suggests that such efforts can help students recognize and develop their agency. Agency, here, means that students use their mathematical knowledge and skills to solve problems germane to their lives" (Stemhagen, 2011)

Stemhagen (2011) is suggesting that problem solving and learning to solve problems helps students to develop agency, which means that the students are learning to develop a voice and feel empowered through their learning experiences. Students need to recognize for themselves that they can communicate, collaborate, and problem solve and use mathematical knowledge. Math class needs to help students to build these skills.

Paul Lockhart (2009), author of The Mathematician's Lament, writes

> "By concentrating on what, and leaving out why, mathematics is reduced to an empty shell. The art is not in the 'truth' but in the explanation, the argument. It is the argument itself which gives the truth its context, and determines what is really being said and meant. Mathematics is the art of explanation. If you deny students the opportunity to engage in this activity— to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs— you deny them mathematics itself. So no, I'm not complaining about the presence of facts and formulas in our mathematics classes, I'm complaining about the lack of mathematics in our mathematics classes." (pg 29)

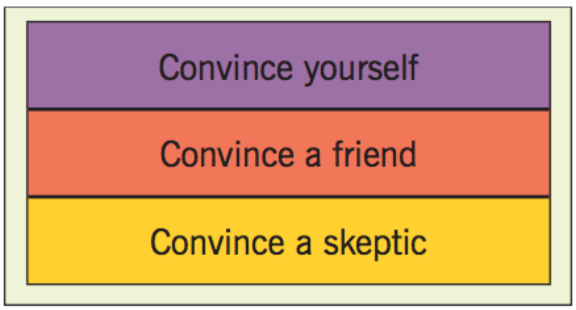
Lockart's (2009) words here describe what is often absent from math classes, the opportunity for real mathematics. Students do not have the opportunity to see the connectedness of math and see the beauty and creative nature of math. Lockhart (2009) goes on to describe how we can teach mathematics, "mathematics is the purest form of art and should be taught as such...Techniques in mathematics, as in any art, should be learned in context. ... Give your students a good problem, let them struggle and get frustrated. See what they come up with. Wait until they are dying for an idea, *then* give them some technique. But not too much." (p. 41-42) Students need to be given opportunities to work through problems that are open ended, require discourse, collaboration, and require some struggle. This is the work of a mathematician. Students need to learn to ask questions, they need to become comfortable masking mistakes, they need to gain comfort talking through their process, and sometimes having to rework their process. Our students need to deepen their mathematical knowledge by investigating a problem and drawing conclusions with the support of a teacher and fellow classmates. Knowledge should be gained via active learning opportunities. Lockhart writes, mathematics is about problems, and problems must be made the focus of a student's mathematical life. Painful and creatively frustrating as it may be, students and their teachers should at all times be engaged in this process ... (of problem solving.) (p. 60)

What does it mean to think like a mathematician? That should be at the forefront of all teachers' minds as they think about instruction and the learning they are planning for their classrooms. Thinking like a mathematician includes:

- Justifying
- Asking questions
- Looking for Patterns
- Drawing Conclusions
- Creating Models
- Making Observations
- Reasoning

This list is not exhaustive, but incorporates many elements of how to think like a mathematician. As a teacher when we think about teaching our students mathematics we need to be teaching how to think like a mathematician. If students are not reasoning in the classroom they are not truly being mathematical and not doing the work of mathematicians. Jo Boaler (2019) explains to students that "scientists prove hypotheses by finding confirmatory or disconfirmatory evidence, mathematicians prove conjectures by reasoning - talking about why methods are chosen, how they work, and how they link to one another, describing the logical connections among them." (Boaler, 2019) Boaler goes on to describe how teachers can support students in learning to reason mathematically. Teachers can provide tasks for students that encourage the students to problem solve as well as reason, but teachers also need to teach ways for students to reason with one another. One example of this is Cathy Humphrey's Skeptic Framework.

This framework asks the students to first convince themselves, then convince a friend, and finally convince a skeptic. This framework on a basic level increases communication in the classroom and it also supports the student's ability to ask and answer questions. A student needs to develop mastery over a problem or prompt to not only be able to convince themselves, but to be able to convince someone that is intentionally questioning their reasoning. Boaler (2019) includes a few questions to support Humphrey's framework: how do you know it works, why did you use that method, and can you prove that use? Questions like this force a student to justify their reason, which aligns to the Common Core Standards of Mathematical Practices. Visual of Cathy Humphrey's Framework:



Cathy Humphrey's Skeptics Framework

Too often in mathematics classes, students work with exercises rather than with problems (Herr, Johnson, & Piraro, 2001). Exercises, as the term suggests, provide opportunities for students to practice a particular skill. Students generally look at the question and know immediately how to arrive at an answer. A real problem (and therefore, a good problem) places students in a more challenging (and compelling) situation, namely needing to determine what to do in the face of not knowing what to do. However, due to their prior experiences in mathematics, most students struggle with this kind of ambiguity. Real problems are not necessarily word problems, although they can be. Too often, curriculum guides equate problem solving with solving word problems. Such problems generally appear at the end of chapters in math textbooks. A real problem is a question to which the answer is not immediately apparent.

Boaler (2013) writes that "mathematical problems that need thought, connection making, and even creativity are more engaging for students of all levels and for students of different genders, races, and socio-economic groups. When there are diverse elements of mathematics encouraged, and it is not just about procedure execution, many more students contribute and feel valued. Boaler refers to this broadening and opening of mathematics taught in classrooms as *mathematical democratization*. We want more of our students to be successful in math classrooms. We need math to be accessible and equitable for our students. For too many the idea that math is too hard, uninteresting, and accessible only to "nerds" still persists. This idea is made even more damaging by harsh stereotypical thinking—mathematics is for select racial groups and men. Boaler (2013) writes how this thinking, as well as the teaching practices that go with it, have provided the perfect conditions for the creation of a math underclass. Narrow mathematics teaching combined with low and stereotypical expectations for students are two of the main reasons that the U.S. is in such a rough place in math education.

According to research done by Jo Boaler, Kasi Allen, Kurt Stemhagen, and other academics, the young people who are successful in today's workforce are those who can discuss and reason about productive mathematical pathways, and who can be wrong, but can trace back to errors and work to correct them. In our new technological world, employers do not need people who can calculate correctly or fast, they need people who can reason about approaches, estimate and verify results, produce and interpret different powerful representations, and connect with other people's mathematical ideas. Allen (2011) writes that

> "in nearly every arena, from personal finance to health care, navigating life in the 21st century requires mathematical thinking, particularly problem solving. Society is increasingly complex and global in nature. Bombarded with information at every turn, we process more in a matter of minutes than many of even our recent ancestors did in their entire lifetime. Never has survival depended so much on the ability to reason logically, to discern facts from fiction, and to make judicious decisions based on the available data. Even careers historically considered low-skill require workers to estimate, to recognize patterns, to reason proportionally, and to use computerized tools that take commands in the form of mathematical statements. All of these situations not only require citizens to think critically but also to communicate their ideas— and here mathematics provides a seldom- recognized opportunity." (Allen, p 3)

The skills that our students need as they graduate high school in order to be a successfully contributing member to our democratic society include the ability to reason, problem solve, communicate effectively, collaborate, and be able to look for patterns amongst all the available information. These are skills that describe the field of mathematics. These are skills are typically absent from most American classrooms.

Allen (2001) writes that democratic mathematics education cannot take place in a classroom where the teacher insists students learn the same way, work toward a single best solution, minimize interaction and teamwork, and focus on the mathematical ends or answers rather than the means or processes. According to the constructivist approach, students should be given the opportunity to discover and construct their own mathematical meanings. Allen (2011) goes on to write that when students are given opportunities to ask their own questions and to extend problems into new directions, they know mathematics is still alive, not something that has already been decided and just needs to be memorized" What if students did not associate mathematics with memorizing algorithms, completing long lists of practice problems, and being judged on the number of correct and incorrect answers? What if, instead, students viewed mathematics as the subject that made them think deeply while seeking the answers to thoughtful questions, many that they came up with themselves? What if math class were something that students looked forward to because they were excited about the challenge of exploring problems and issues that they value as important and relevant? What if students were excited to come to math because they were excited about the challenge of continuing to explore a problem that they hadn't been able to find the answer to the previous day?

Ellis and Malloy (2007) propose a framework for democratic mathematics classrooms that extends beyond constructivism and moves in the direction of mathematics as thinking. They suggest four key elements to define such classrooms: (a) problem- solving curriculum, (b) culture of inclusiveness and rights, (c) equal participation in decisions, (d) equal encouragement for success. In such classrooms, students work collaboratively to solve problems that they value as important in their lives. Building on their prior and collective knowledge, students develop the skills to locate relevant information when they need it and to use multiple representations to gain new insights. Students must have the opportunity to regularly communicate their ideas through writing and speech; they must feel they have a say in their own learning and in the direction of classroom discourse. This framework can serve as a guide to how we create equitable learning spaces that offer students opportunities to learn math authentically as an active learner.

Teaching Strategies

Math Talks

Are a teacher-led student-centered technique for building math thinking and discourse. First the Teacher presents the problem. Computation based problems should always be presented horizontally to discourage the focus on the use of a standard algorithm. Next the Students figure out the answer. The students are given time (1–2 minutes) to silently and mentally figure out the answer. Part of the directions should be that this is an activity done without pencil and paper. Students can then signal quietly to the teacher (e.g. with a thumb up against their chest) when they have an answer. Students share their answers. And a few students volunteer to share their answers and the teacher records them on the board. The teacher records all answers that students arrived at without judgement. Students share their thinking. Students share how they got their answers to the group. The teacher records the student's thinking and attaches their name to the solution. As the students are sharing their thinking, the teacher asks questions that help them express themselves, understand each other, and clarify their thinking to make sense of the problem and its solution(s). Multiple ways of solving problems and the connections among them are emphasized.

Probing questions (to help students express their thinking)

- Can you tell us where you got that?
- How did you figure that out?
- What was the first thing your eyes saw, or your brain did?

Connecting questions (help students respond to each other's thinking)

- Who did it another way?
- What questions do you have for them?
- Do you agree or disagree? Why?

Notice & Wonder
What: This routine can appear as a warm-up or in the launch or synthesis of a classroom activity. Students are shown some media or a mathematical representation. The prompt to students is "What do you notice? What do you wonder?" Students are given a few minutes to think of things they notice and things they wonder, and share them with a partner. Then, the teacher asks several students to share things they noticed and things they wondered; these are recorded by the teacher for all to see. Sometimes, the teacher steers the conversation to wondering about something mathematical that the class is about to focus on.

Where: Appears frequently in warm-ups but also appears in launches to classroom activities.

Why: The purpose is to make a mathematical task accessible to all students with these two approachable questions. By thinking about them and responding, students gain entry into the context and might get their curiosity piqued. Taking steps to become familiar with a context and the mathematics that might be involved is making sense of problems (SMP1).

Low Floor/High Ceiling Tasks:
A task that allows everyone to get started, but also allows everyone to get stuck or challenged. The task needs to be accessible for all students, and all learners can

potentially reach a point where they don't immediately know what to do next, and they can start to develop their perseverance

Questioning

Here is a list of questions from the Professional Standards in Teaching Mathematics, grouped into categories that reflect the mathematical practices.

- *Helping students access the problem:*
    - What do you already know about this problem?
    - Can you draw a picture of the situation?
    - Have you ever solved a problem like this before?

- *Helping students work together to make sense of mathematics:*
    - What do others think about what Janine said?
    - Do you agree? Disagree?
    - Does anyone have the same answer but a different way to explain it?
    - Do you understand what they are saying?

- *Helping students to rely more on themselves to determine whether something is mathematically correct:*
    - Why do you think that?
    - Why is that true?
    - How did you reach that conclusion?
    - Can you make a model to show that?

- *Helping student learn to reason mathematically*
    - Does that always work?
    - Can you think of a counterexample?
    - How can you prove that?
    - What assumptions are you making?

- *Helping students learn to conjecture, invent, and solve problems:*
    - What would happen if…? What if not?
    - Do you see a pattern?
    - What is alike and what is different about your method and her method to solve the problem?
    - Can you predict the next one? What about the last one?

- *Helping students to connect mathematics, its ideas, and its applications:*
    - How is this process like others that you have used?
    - How does this relate to ___________?
    - Have you ever solved a problem like this before?
    - Can you give me an example of _________?

3 Read Protocol

The Three Read Protocol is one way to do a close read of a complex math word problem or task. This strategy includes reading a math scenario three times with a different goal each time. The first read is to understand the context. The second read is to understand the mathematics. The third read is to elicit inquiry questions based on the scenario.

First Read: Teacher reads the problem stem orally. *Key Question: What is this situation about?*

Second Read: Class does choral read or partner read of the problem stem. *Key Question: What are the quantities in the situation?*

Third Read: Partner or choral read the problem stem orally one more time. *Key Question: What mathematical questions can we ask about the situation?*

Classroom Activities

Each of the five classroom activities described in this unit can be taught independently. Two of the activities in this unit are connected to an Algebra 1 unit on Systems of Equations and could be used at some point during that time period. The other three do not have a specific unit that they connect to. This curricular unit is designed to support teachers incorporating task-based learning into their classrooms. Each activity is built around a common core standard for mathematical practice. The goal of the activity is to support the development of the habits of a mathematician. The goal being, that the students begin to develop comfort with thinking like a mathematician. These activities can be done like a math lab, where a teacher creates a space in their week where they block off specific time once a week, where they have students work on a challenging task that does not always have a specific answer.

Activity #1

Lesson Title	The Art of Problem Solving: Systems of Equations
Lesson Length	2 Days - 1 day for the students to work on the assignment, 1 day to discuss in class and troubleshoot mistakes and discuss questions

Common Core Content Standards	HS.A-REI.C.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
Common Core Standards for Mathematical Practice	SMP #1. Make sense of problems and persevere in solving them
Lesson Narrative	Students working with solving systems of equations starts the same way as working with solving equations in one variable; with an understanding behind the various techniques. An important step is realizing that a solution to a system of equations must be a solution [to] all of the equations in the system simultaneously. Then the process of adding one equation to another is understood as "if the two sides of one equation are equal, and the two sides of another equation are equal, then the sum of the left sides of the two equations is equal to the sum of the right sides." Since this reasoning applies equally to subtraction, the process of adding one equation to another is reversible, and therefore leads to an equivalent system of equations.
Lesson Activities	***Inclusive Student Activity:*** From the Art of Problem Solving: Tweedledum says, “The sum of your weight and twice mine is 361 pounds.” Tweedledee says, “Contrariwise, the sum of your weight and twice mine is 362 pounds.” If they are both correct, how much do both Tweedledum and Tweedledee weigh together? *Key Concept*: The focus with this question is that the students do not necessarily have to solve for the variables individually to solve this problem. It is important to make sure you know what you’re looking for in a problem, so you know when you’ve found what you need. Substitution or elimination can be used to work through this problem, but the students might recognize the similar forms of the two equations and try adding the two equations together. This leaves them with $3x + 3y = 723$. They could then divide everything by 3, and are left with $x + y = 241$, which would be the sum of their weight.

	Screens 11 & 12 on the Systems of Equations Practice on Desmos are from the Art of Problem Solving and even with virtual learning have the students work with a partner. In class that wouldn't be an issue, but virtually the students were encouraged to use facetime, google meet, zoom, or whatever app felt comfortable for them to meet with a partner so that they could connect with a classmate to work on these two problems. Part of the question prompt asked them to include who they worked with and what they and their partner discussed. https://teacher.desmos.com/activitybuilder/custom/603907c62af3213a00bfab5e
Checks for Understanding/Assessment	This practice includes varying entry points of systems of equations and solving equations, it is a good formative assessment for students and teachers to check for student understanding

Activity #2

Lesson Title	Solving Systems by Elimination Part 1, part of the Illustrative Math Algebra 1 Curriculum
Lesson Length	1 - 2 Days
Common Core Content Standards	**Addressing** HSA-REI.C.6: Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. **Building Towards** HSA-REI.C.5: Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
Common Core Standards for Mathematical Practice	SPM#3 - Construct viable arguments

Lesson Narrative	Students warm up to the idea of adding equations visually. They examine a diagram of three hangers where the third hanger contains the combined contents of the first two hangers and all three hangers are balanced. Then, they analyze the result of adding two linear equations in standard form and notice that doing so eliminates one of the variables, enabling them to solve for the other variable and, consequently, to solve the system. In studying and testing a new strategy of adding equations and then offering their analyses, students construct viable arguments and critique the reasoning of others (SMP3)
Lesson Activities	***Opening Routine***: Notice & Wonder Hanger Diagrams Have students write: two things that they notice about the hanger diagram and write at least one question that they have about the hanger diagrams. Give students about a minute to a minute and a half of quiet think time and then allow for student sharing. If using this lesson on desmos, can use the anonymize feature and share student responses via screen share. After all responses have been recorded without commentary or editing, ask students, “Is there anything on this list that you are wondering about now?” Encourage students to respectfully disagree, ask for clarification, or point out contradicting information. The idea to emphasize is that the weights on each side of the third hanger come from combining the weights on the corresponding sides of the first two hangers. If no one points this out, raise it as a point for discussion. ***Formative Task: Adding Equations Task*** In this activity, students continue to develop the idea of adding two equations to form a third equation and use it to help them solve systems of linear equations. Along the way, students examine the work of others and practice explaining their reasoning and critiquing that of others (SMP3). They also see that sometimes adding equations is a productive way to solve systems, but other times it isn't. ***Guided Instruction:*** Help students improve their writing by providing them with multiple opportunities to clarify their explanations through conversation. “How did you use Diego’s method to solve this problem?” or “Can you say more about...?”

	Inclusive Student Activity: Adding & Subtracting Equations to Solve Systems This task is an extension of the lesson from the day. It is an opportunity for students to explore using graphing technology. For students that are struggling to add or subtract the equations the teacher can provide the equations to them already in slope intercept form so that the students can see the graphs on the graphing technology. In this task, the students are now being asked to graph each pair of equations in the systems given earlier, as well as the third equation that came from adding or subtracting those equations, and then make some observations about them. The goal observation being that the graph of the third equation intersects the other two graphs at the exact same point—at the intersection of the first two **14.3: Adding and Subtracting Equations to Solve Systems** Here are three systems of equations you saw earlier. System A: $\begin{cases} 4x + 3y = 10 \\ -4x + 5y = 6 \end{cases}$ System B: $\begin{cases} 2x + y = 4 \\ x - y = 11 \end{cases}$ System C: $\begin{cases} 8x + 11y = 37 \\ 8x + y = 7 \end{cases}$ For each system: 1. Use graphing technology to graph the original two equations in the system. Then, identify the coordinates of the solution. 2. Find the sum or difference of the two original equations that would enable the system to be solved. 3. Graph the third equation on the same coordinate plane. Make an observation about the graph. Reflective Closure/Check For Understanding: What to do with this system

	As a way to check the students understanding a formative assessment task can be used where a system is given to the students with the prompt: Here is a system of linear equations: $\begin{cases} 2x + \frac{1}{2}y = 7 \\ 6x - \frac{1}{2}y = 5 \end{cases}$ 1. Which would be a more helpful for solving the system: adding the two equations or subtracting one from the other? Explain your reasoning. ***Link to Lesson on Desmos:*** https://teacher.desmos.com/activitybuilder/custom/60156eabeee77f37bf0c9efd

Activity #3

Lesson Title	Math Talk - Regrouping Fractions
Lesson Length	This lesson activity is an example of an opening routine that should take about 10-15 minutes at the start of class.
Common Core Content Standards	Building On: **4.NF.B** Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers. Addressing: **HSN-RN.A.1** Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5. **HSN-RN.A.2** Rewrite expressions involving radicals and rational exponents using the properties of exponents.
Common Core Standards for Mathematical Practice	SMP#7 - Look for & Make use of Structure SPM#3 - Construct viable arguments
Lesson Narrative	The purpose of this Math Talk is to elicit strategies and understandings students have for decomposing fractions into a unit fraction times a whole number and using the associative and commutative properties. These understandings help

	students develop fluency and will be helpful later in this lesson when students will need to be able to decompose a rational exponent into a unit fraction times a whole number.
Lesson Activities	Opening Routine: Math Talk Display one problem at a time. Give students quiet think time for each problem and ask them to give a signal when they have an answer and a strategy. Keep all problems displayed throughout the talk. Follow with a whole-class discussion. Ask students to share their strategies for each problem. Record and display their responses for all to see. To involve more students in the conversation, consider asking: • "Who can restate _____'s reasoning in a different way?" • "Did anyone have the same strategy but would explain it differently?" • "Did anyone solve the problem in a different way?" • "Does anyone want to add on to _____'s strategy?" • "Do you agree or disagree? Why?" Find the value of each expression mentally. $\frac{1}{2} \cdot 5 \cdot 4$ $\frac{5}{2} \cdot 4$ $\frac{2}{3} \cdot 7 \cdot \frac{3}{2}$ $7 \cdot \frac{5}{3} \cdot \frac{3}{7}$ Screen 2 on the Desmos Lesson - https://teacher.desmos.com/activitybuilder/custom/606a58ba5354291d5fd5162e

Activity #4

Lesson Title	The Seven Bridges of Konigsberg
Lesson Length	2 - 3 days
Common Core Standards for Mathematical Practice	SMP #1. Make sense of problems and persevere in solving them SPM#3 - Construct viable arguments
Lesson Narrative	This question was given to a famous mathematician called Leonhard Euler. The town of Königsberg straddles the Pregel River. It was formerly in Prussia, but is now known as Kaliningrad and is in Russia. Königsberg was situated close to the mouth of the river and had seven bridges joining the two sides of the river and also an island and a peninsula. It seemed that no matter where they started or which way they went, the townspeople could not find a route that used all seven bridges exactly once. Finding such a route became known as the Seven Bridges of Konigsberg Problem. In 1736, the Swiss mathematician Leonhard Euler proved that there was no route in Konigsberg that used all seven bridges exactly once. He reinterpreted the problem by changing all the islands to points and all the bridges to line segments. Euler then reasoned two things for a continuous (or walkable) path to exist in such a system: 1) Every point except for the starting and ending points must have an even number of line segments connected to it (because it would need n times to enter and n times to exit for a total of 2n). 2) There can be at most two points that have an odd number of line segments connected to it – the starting and ending points. In the Konigsberg problem, all four points have an odd number of line segments connected to it (the northern island is connected by 3 bridges, the western island is connected by 5 bridges, the eastern island is connected by 3 bridges, and the southern island is connected by 3 bridges), so a solution is impossible.

	The goal of this is for the students to work on a problem and come up with possible solutions just like Euler did. The students need to try and fail and then try again. As a class the students can discuss, what they tried and what they discovered. From there as a class you can discuss what discoveries this problem led to and the branches of mathematics it led to.
Lesson Activities	Opening Routine: ***Formative Task***: A river flows through the middle of Konigsberg, forming an island in the middle and then separating into two branches. The citizens of Konigsberg have built seven bridges to get from place to place. The people wondered if they could walk around the city in such a way that they would cross each bridge once and only once. ***Guided Instruction:*** Brief History about Leonard Euler and the history of Konigsberg. Demonstrate and work through the solution that there must be an even number of bridges ***Inclusive Student Activity***: Build a mobius Strip. Using a 24'x 2' inch piece of white paper, ask students to draw a line down the center of the piece of paper. Then invite the students to decorate the sides using symbols, patterns, and colors. Ask students what they think will happen if they cut their paper strips in half by cutting along the centerline. Discuss the possibilities and the reasons for their suggestions. Then have them cut along the center line and see what happens. Are the results what they expected? What do they think will happen if they cut what they have in half again? Have the students try it. ***Reflective Closure:*** Discuss how Euler's discovery led to the fields of graph theory and topology. Lesson Activity Built Into Desmos: https://teacher.desmos.com/activitybuilder/custom/60ab9b27b72987080dde7df7

Activity #5

Lesson Title	Checkerboards
Lesson Length	1 Day

Common Core Standards for Mathematical Practice	SPM #8 - Look for & express regularity in repeated reasoning
Lesson Narrative	This activity allows students to explore number patterns they make on a grid. By changing the size of the grid and the number they count, students discover different patterns that are made on the grid. After exploring with pictures they begin to generalize and make conjectures as to how to create different patterns. Exploring and generalizing through visuals are an important aspect of mathematics
Lesson Activities	***Opening Routine***: A. Pre-Survey question to students about how they value struggle in terms of their academics. B. Play the mindset video, The importance of Struggle ***Formative Task:*** *Checkerboard Activity Part 1* Look at the hundreds chart for a few different patterns. Describe two patterns below, using numbers from the chart in your description. *Checkerboard Activity Part 2* This checkerboard was created by counting by twos. Look for a few patterns in the shaded diagonals. Describe two of them, using numbers from the chart in your description. ***Inclusive Student Activity:*** *Create your own Checkerboard* Create your own checkerboard pattern by writing on the chart on the left. Describe one of the patterns in the shaded diagonals using numbers from the chart in your description. ***Reflective Closure/Check For Understanding:*** Remind students of the video messages they heard What patterns do you notice today? What patterns did you create today? Invite students to share their strategies that they used to create these patterns.

	Share with students that part of being a mathematician involves exploring problems, finding patterns and making conjectures. ***Link to Lesson on Desmos:*** https://teacher.desmos.com/activitybuilder/custom/5f3429196ca75f283ae82559

Lesson Title	Broken Eggs
Lesson Length	1 Day
Common Core Standards for Mathematical Practice	SPM #1 - Make sense of problems & persevere in solving them
Lesson Narrative	The overarching goal of this assignment is for students to be able to make sense of a situation and use multiple approaches to justify their answers. Students will be able to explain their mathematical thinking in writing. This assignment might take multiple iterations, it might take making some mistakes (remember your brain is growing), and it will definitely take your critical thinking and problem-solving skills. The goal of this assignment is not just to get the correct answer. The goal is to be able to explain your process, to walk us through how you got to that answer, and to be able to reflect on the question. Students work on a math riddle. This lesson gives them the opportunity to solve a problem in different ways and spend some time explaining their own, unique mathematical thinking. The Broken Eggs problem is a low-entry / high ceiling problem that allows students multiple entry points and ways to solve. This problem is derived from a well-known problem written by the Hindu mathematician Brahmagupta and comes out of the CPM curriculum, but I adapted it to make it more locally and culturally relevant for the students of Philadelphia.
Lesson Activities	Opening Routine: Three Reads Strategy. Begin class by having students read the Broken Eggs problem aloud together. First just

	for context of the situation, then looking for the numbers given, then asking for what is being asked. ***Formative Task:*** Let students get to work! If they have trouble starting, I like to use manipulatives to illustrate the problem. You can have a bunch of small objects (I use pennies) and have them represent the eggs. You can ask students to choose a number of eggs to start with. See if anyone has ideas about what number might make sense. If students say 8 eggs for example, you might try and see if anyone can rule out the number 8 because it is divisible by 2 and therefore does not have a remainder. You may also need to help students come up with a way to organize their information. You might suggest some kind of chart where they are testing numbers and what the remainders are. Students may begin to notice a pattern within the chart. Remind students to keep notes about different strategies that they try as they work. Reinforce that you want to hear about strategies that both were successful and not successful. That is all part of the process. Something else that will be new to the students is you want to hear about who they talked to while working on this problem. Collaboration and communication are important parts of the problem. If a student asked an older peer or a sibling or a parent and they worked through the problem, that becomes part of their process. Students need a reminder that all of these pieces end up in their write up. Students may begin to get exasperated when the numbers start to get really big. This is where you can help them with **SMP1: Make sense of problems and persevere in solving them.** You can prompt students with questions about ways to get unstuck or if there are tools other than manipulatives that might be helpful. ***Link to Lesson:*** ***[Broken Eggs Task]***

Resources

Reading List

Allen, K. (2011). Mathematics as thinking a response to “democracy and school math.” democracy & education, 19(2). Retrieved from: https://democracyeducationjournal.org/cgi/viewcontent.cgi?article=1036&context=home

> Kasi Allen’s article provided background research on the need to change the way that we teach math so that we can better prepare students for the skills they will need to enter the job force. Allen also reinforces that math should be taught in a democratic approach as a thought-based subject. Students need to be thinking for themselves in the subject, and need to realize that they subject is still alive and happening.

Boaler, J. (2018). Developing mathematical mindsets: the need to interact with numbers flexibly and conceptually. *American Educator*.

> This article by Boaler talked about the importance of developing a growth mindset in students and why it is important to offer open ended opportunities for students are allowed to explore with numbers and concepts.

Boaler, J. (2013, Nov 12). The Stereotypes That Distort How Americans Teach and Learn Math. The Atlantic. Retrieved from https://www.theatlantic.com/education/archive/2013/11/the-stereotypes-that-distort-how-americans-teach-and-learn-math/281303/

> This article provided background knowledge on the current status of math education in the United States. It also provided information on the stereotyping that exists among math education and the important of offering an equitable education.

Boaler, J., Chen, L., Williams, C., & Cordero, M. (2016). Seeing as understanding: the importance of visual mathematics for our brain and learning. Journal of Applied & Computational Mathematics 5 (5)

Boaler, J. (2019). Prove it to me! Mathematics Teaching in the Middle School 24 (7) 422-429.

This article was important in talking about student discourse and having the students be able to justify their work. This article also introduced me to Cathy Humphrey's skeptics framework.

Boaler, J (1998). Open and closed mathematics: student experiences and understandings. Journal for Research in Mathematics Education Vol. 29, No. 1, 41–62

This resource allowed me to further differentiate between open mathematics and closed mathematics and what closed mathematics actually means in terms of the classroom.

Boaler, J., Dieckmann, J., Pérez-Núñez, G., Liu Sun, K. & Williams, C. (2018). Changing students minds and achievement in mathematics: the impact of a free online student course. Front. Educ 3:26.

Boaler, J. (2016). Mathematical Mindsets, Jossey-Bass.

Boaler's book provided a lot of the background research on why math class is treated differently and the importance of mistakes and productive struggle. Boaler is an acclaimed academic, professor, and research in the field of math education.

Boaler, J. & Williams, C. (2015). Fluency without fear: research evidence on the best ways to learn math facts. You Cubed at Stanford University.

This resource provided some of the background information about where math anxiety comes from.

Dewey, J. (1910). How we think. Lexington, MA: D.C. Heath

John Dewey was briefly referenced in this curriculum. Dewey believed in active learner and empowering students through their learning experiences.

Dweck, C.S. (2006a) Mindset: the new psychology of success. New York: Ballantine Books.

Carol Dweck's books was important because it provided the research on mindsets. Part of the brain research of this unit is that many students come in to math class with a fixed mindset. Dweck talks about a fixed versus growth mindset and how as teachers we can best support our students.

Ellis, M., & Malloy, C. (2007). Preparing teachers for democratic mathematics education. In D. Pugalee, A. Rogerson, & A. Schinck (Eds.), Proceedings of the Ninth International Conference: Mathematics Education in a Global Community (pp. 160- 164). Charlotte, NC.

This resource was useful in thinking about how we need to adapt our current math classrooms. What skills will our students need to be successful in a democratic society, and how as teachers can we prepare to teach to support our students effectively.

Lehoczky, S., & Rusczyk, R. (2015). *The art of problem solving: introduction to algebra.* AoPS Inc.

The Art of Problem Solving was used for one of the lesson activities in this unit. This entire book is filled with activities that can be used in the classroom on math lab days. Problems that the students can work on that encourage questions, multiple representation, collaboration, and communication, generalizing, pattern seeking, and other habits of a mathematician.

Lockhart, P. (2009). A mathematician's lament. New York, NY: Bellevue Literary Press

Paul Lockhart talked about what is and what is not in math classrooms. Lockhart examines what mathematics really is and explores why true mathematics is missing from math classrooms.

Moser, J., H. S. Schroder, C. Heeter, T. P. Moran & Y. H. Lee. (2011). Mind your errors: evidence for a neural mechanism linking growth mindset to adaptive post error adjustments. Psychological Science 22: 1484–9.

This research furthered my knowledge about what happens in the brain when one makes a mistake and the changes that happen with different learning activities.

NCTM. (2014). " Effective Teaching and Learning". In Principles to Actions: Ensuring Mathematical Success for All. Reston VA, USA: The National Council of Teachers of Mathematics, Inc.. Retrieved Jun 21, 2021, from https://pubs.nctm.org/view/book/9780873539043/c02.xml

Principles to Action provided critical information about what productive struggle looks like from the teacher and student view point. It also provided important information about what NCTM defines as effective teaching practices.

O'Donnell, A. (2014). Another relationship to failure: reflections on Beckett and education. Journal of Philosophy of Education, Vol. 48, No. 2

This resource was useful in rethinking how we define failure. Students often think of themselves as the failure when they fail, but O'Donnell proposes, separating oneself from the action. This resource also helped think about why success is so often a comparison to others, as opposed to thinking about your own measures of performance. This leads to the conversation of the performance-based culture that we living in and even more specifically how math class is so performance driven.

Ray, M. (2013). Powerful problem solving: activities for sense making with the mathematical practices. Heineman, Portsmouth, NH.

Max Ray's problem-solving book is a book of possible activities to use in the classroom as a teacher thinks about assigning tasks to their students. These are strong problems that do not have single answer or do not have a single method of solving. This book is also a resource talking about the importance of productive struggle.

Selbach-Allen, M. E., Williams C. A., & Boaler, J. (2020). What Would the Nautilus Say? Unleashing Creativity in Mathematics! Journal of Humanistic Mathematics. 10 (2), 391-414.

Math is a subject of creativity. It is a subject built around reasoning and human interaction, yet that is almost never present in math classrooms. This article was a resource to help delve into this topic.

Sriram, R. (2020, April 13). The neuroscience behind productive struggle. Retrieved from:
https://www.edutopia.org/article/neuroscience-behind-productive-struggle

This resource was very helpful in defining the brain science of learning. It described the neurological processes. It also went further to describe why productive struggle increases brain growth.

Stemhagen, K. (2011). Democracy and School Math: Teacher Belief-Practice Tensions and the Problem of Empirical Research on Educational Aims. *Democracy and Education*, 19 (2), Article 4. Retrieved from:
https://democracyeducationjournal.org/home/vol19/iss2/4

This resource was useful because Stemhagen did research around math education and looked at the skills necessary for 21st century jobs. Stemhagen also explored the connections between democracy and math classrooms, and what that looks like.

Stigler, J.W. & Hiebert, J. (1999) The Teaching Gap. New York: Free Press

This resource provided a comparison of math class in the United States to other countries. And allowed us to question why the United States is so different?

Standards for Mathematical Practice

"Standards for Mathematical Practice." *Standards for Mathematical Practice | Common Core State Standards Initiative*, www.corestandards.org/Math/Practice/.

This resource is the backbone of the math content for the unit. The standards for mathematical practice outline the habits and practices that the students need to demonstrate within the classroom.

Appendix

HS.A-REI.C.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

HSA-REI.C.6: Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

SPM #1 - Make sense of problems & persevere in solving them

SPM#3 - Construct viable arguments

SMP#7 - Look for & Make use of Structure

SPM #8 - Look for & express regularity in repeated reasoning

A description of how your unit implements the academic standards