Visualizing Differential Calculus for the IB Student

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Unit Prospectus

I teach math at the Philadelphia High School for Girls. The first public high school for women in the USA was the Girls' Normal School, founded in 1848, to prepare women to become teachers. In 1893, the Girls' Normal School separated into two schools and the Philadelphia High School for Girls was formed in order to prepare young women for college. Girls High, as it is colloquially called, attracts young women from all parts of the city and has a diverse student body. To this day, college preparation is the main focus of the school. (https://girlshs.philasd.org/about-us/). Many of the students take Advanced Placement courses in areas where they excel. Many well-rounded students who do well in all areas pursue the International Baccalaureate Diploma.

The International Baccalaureate (IB) Diploma Program is a world-wide program that focuses on creating globally-minded learners. It offers rigorous academic courses with an international focus. Students are encouraged to ask questions and discover the answers on their own. The home page of the IB website states:

> When we talk about education and what it means, what we really want to know is how to instil (sic) in our students the knowledge that will make them better learners and better people. How can we be sure that we send them off into life with the skills they really need to grow and develop in a successful, happy way? Now in our 52nd year, we're more dedicated than ever to developing international education that creates a better world. (https://www.ibo.org/about-the-ib/)

Students admitted into the International Baccalaureate Program at the Philadelphia High School for Girls are generally described as "good students." Prior to applying to the program, they attend a presentation describing the program's benefits and requirements, including the fact that this rigorous program will require many more hours of work than the typical high school curriculum. The program consists of a full schedule of classes during the junior and senior years. Unlike other high students who choose their classes and apply for honors and Advanced Placement level classes based on their skills, IB students may only choose which foreign language to study. The rest of their courses are predetermined. The benefit of this system is that the students get to know each other very well and tend to be more supportive of each other than students who may share just a class or two on their schedules. The challenge is that students of varying abilities make up the IB classes. If the only students accepted into the program were those who excelled in every academic discipline, then there would not be a large enough enrollment to support the program. My IB math students vary in ability from gifted to struggling.

Unlike the other IB courses, my math class is made of both juniors and seniors. Generally, those that are eligible to take IB math during their junior year have more advanced math skills; they either passed Algebra 1 in eighth grade or performed well enough in Algebra 1 to take both Algebra 2 and Geometry during their sophomore year. Seniors in my IB math class followed the traditional route of just one math class per year and tend to have less advanced mathematical skills than their junior-year classmates.

In order to remain relevant and current, IB reviews its curriculum every seven years and the curriculum is changing next year. While the number of topics has been reduced, more material has been added than deleted. The calculus curriculum especially worries me. In the course I currently teach, Math Studies, students are introduced to calculus by differentiating polynomials and applying that knowledge to finding the equations of tangent and normal lines at given points and solving optimization problem. My classes have managed the calculus unit well and most of them have gone on to pass their comprehensive IB exam in May. Next year, Math Studies will be replaced by Mathematics: Applications and Interpretation. While the differential calculus unit is very similar, the students will also learn to use integration to find area. The suggested number of hours as provided on the International Baccalaureate Organization's website has increased from eighteen hours (https://resources.ibo.org/data/d_5_matsd_gui_1203_1_e.pdf) to nineteen hours. (https://resources.ibo.org/dp/subject/Mathematics-2021-applications/works/dp_11162-53993?root=1.6.2.8.5.3&lang=en) Unwilling to teach integration in one hour, I need to reduce the number of hours teaching differentiation. Differentiation will be the focus of the unit I am presenting here, although I hope to develop similar units on integration and other topics I teach for my own classroom use.

I will need to teach effectively and efficiently in order to present all of the content. I welcome my seminar: A Visual Approach to Learning Math. Calculus is an intimidating subject to many students. Bringing Professor Ghrist's influences on art and animation into the presentations will most likely make the subject more attractive and understandable to at least most, if not, all of my students. Increasing my students' engagement will help to make the time in my classroom more effective. Finally, IB students tend to be more well-rounded than the average student so the artistic addition to instruction should appeal to them. Although all of my students are female, studies have shown that there is no statistical difference of visual preference between genders. (Haciomeroglu & Chicken, 2012)

The animation we have explored in the seminar has been created in PowerPoint so that is the format for my project. The project use animation to illustrate how functions change as we move along their curves, illustrating the change in the slope of the tangent line, for example. Links to videos explaining how to animate within SMART Notebook are included for those who prefer to use SMART rather than Power Point.

My students have historically done well finding the tangent at given points and finding the equation of the tangent line…. then I introduce the normal line and their understanding and comprehension give way to confusion. My expectation is that attractive illustrated diagrams will be much more memorable than straightforward formulas and note printed in black on a plain white background.

After presenting notes and definitions in the first two sections, the chapter's third section focuses on applying those concepts to optimize values. This unit contains illustrations of example problems so students have a picture to accompany the words in their textbooks. They should serve as a guide for the students and help them develop their own models, diagrams, and illustrations.

A note about the example problems in this unit: the problems solved and/or illustrated in the attached slides are ones that I created. Reasons that I chose to not use the problems or illustrations in the textbook include my not wanting to use copyrighted material, to offer further examples, and to make this lesson appropriate for use with other textbooks. The textbook that I use in my IB classes contains detailed solutions for all example problems as well as answers for all the problems in the exercises. Students and teachers can and should review the problems in the textbook used in their own classrooms.

Content Objectives

Despite the different levels and abilities, all of my students must master the same curriculum and pass the same comprehensive exam at the end of the year. Students who do not earn enough total points on their comprehensive exams do not earn the special IB diploma that they have strived for two years to earn. As their teacher, I must devise ways to make the material understandable but also memorable so that when they sit for their exam in May, they can recall and apply what they have learned all year. The primary objective of this unit is that students will be able to develop an understanding of differential calculus in order to apply their understanding and solve problems on their IB exam. The IB calculus unit may be further broken into the following objectives:

Students will be able to analyze the rate of change over intervals in a function. The class will begin with linear functions where they can readily see that the slope measures the rate of change and is constant along the entire line. They will next calculate the slope of the secant line passing through the initial and terminal points of a given interval on a parabola. The intervals will be shorter and shorter, leading the class to an understanding of instantaneous rate of change and how it relates to the slope of the line drawn tangent to the curve at the initial point. After they have made that connection, they will be presented with the definition of derivative. Throughout the study of differential calculus, I will remind the class of the relationship between the derivative, the slope of the tangent, and the instantaneous rate of change.

Students will be able to find the equations of the tangent line and the normal line at a given point along a curve. Students will extend their knowledge that the slope of the tangent line is the derivative of the function evaluated at a given value of x. They will synthesize that concept with their prior knowledge of finding the equation of a line, given a point and the slope. After mastering that skill, the class will recall that the slopes of perpendicular lines are opposite reciprocals and find the equation of the line perpendicular to the tangent line at the given point. They will call this perpendicular line the normal line as they move from vocabulary used in Algebra and Geometry to vocabulary used in calculus.

` Students will solve real-life problems involving tangent and normal lines. The IB exam is not comprised of a set of short, direct problems. Rather, it requires the test-takers to apply their skills to solve word problems. Therefore, after knowing how to calculate the equations of the tangent and normal lines, students will practice problems, involving situations such as bicycling, designing a machine, and landscape design. (Belcher, 2019)

Students will be able to identify stationary points. After finding the derivative of a function at several points along a curve, the class will be guided to discover that the derivative of a function equals zero at local minimum and maximum points. Students will find the derivative of a function and solve for the values of x at which the derivative equals zero. Locating the points with those x-coordinates, students will identify each of those points and the endpoints of the stipulated domain as a local maximum, local minimum, global maximum, or global minimum.

Students will determine intervals when the derivative is positive (the function is increasing), negative (the function is decreasing), or equal to zero (stationary points, points of inflection, or horizontal intervals). They will use changing signs to identify local maximum and minimum points.

Students will solve problems involving maximum and minimum points. The IB curriculum requires application of skills rather than mere practice of traditional problems. Students will use their ability to find maximum and minimum points to solve problems related to projectile motion and maximizing profits. (Belcher, 2019)

Students will solve optimization problems. The majority of optimization problems that IB students solve involve geometric figures. Examples include minimizing the surface area of a container while maintaining constant volume, maximizing area for a given perimeter, and determining the number of units a company must sell in order to maximize profit. (Belcher, 2019)

Teaching Strategies

The strategies presented here are techniques that will enable students to actually visualize their lessons. Rather than the typical black print on a white screen or non-detailed line drawings, concepts are presented using color, animation, and manipulation of graphs. The students will be guided to discover concepts and the vivid illustrations will help them to recall information.

The first activity of the unit will be to graph lines and describe how the value of the slope relates to the position of the line, that is how the sign of the slope affects the direction of the line and the magnitude determines the steepness. They will next graph parabolas along with the secant line drawn through the initial and terminal points of a given interval. They will also graph the line tangent to the curve at the initial point. Using Desmos online graphing calculator, they will observe how the slope of the secant line changes as the terminal point moves closer to the initial point. Ideally, they will all discover the slope of the secant lines becomes closer to the slope of the tangent line as the length of the interval shortens.

To find the equations of the tangent and normal lines, students will follow a particular set of steps. Much of this lesson involves writing and remembering definitions and procedures. Inspired by the comic book styled slides from my seminar, I will present this information in a similar style. The colorful layout will help to make the concepts memorable and allow me to present the information in a concise manner, augmenting a limited number of words with illustrations. I imagine my students happily sketching in their notebooks, copying or improving upon my sketches rather than hurriedly copying bland notes. Practicing this lesson will be presented on the graph of a function, allowing the students to see precisely what the tangent and normal look like at points along the function.

As the students begin applying the concepts of tangent and normal lines, they will work with illustrations and gradually move into word problems. Throughout my teaching career, I have always encouraged students to begin most word problems with a sketch. The more artistic students easily accomplish this but some struggle to connect the words in the problem with an illustration of what is happening in the problems. In this unit, I will begin with illustrations and work towards words so that by the time they read a word problem, they are inspired by the pictures they have been viewing on my Smartboard and sketching in their notebook.

The lesson on stationary points, the derivative at those points, and finding local minima and maxima will be a hands-on discovery lesson. They will begin by sketching graphs by hand and drawing tangent lines at assigned points. Recalling the significance of magnitude and direction of the slope, they will identify intervals of their graph where the slope is positive and negative. They will also mark the turning points, that is, the local extrema. Next, they will find the derivative of the function and graph the derivative on the same grid as they graphed the function. After identifying where the derivative is positive, negative, and equal to zero, they will compare the marked intervals on the graphs of the original function and its derivative. In groups, they will discuss their findings and make a conjecture about how the derivative of a function can be used to find local minimum and maximum points. A PowerPoint presentation will follow that illustrates how to find the x-values where the derivative is equal to zero while noting the sign of the derivative before and after such values. The PowerPoint will be similar to those described above with comic book style used for presenting notes and moving, morphing illustrations of the concepts. Movements will involve the graph of the function's derivative evolving from the original graph of the function. Points where the derivative equals zero will be placed in vertical alignment with the local extrema. Intervals where the function is increasing will be highlighted and labeled as positive and graphed as positive values of the derivative. Intervals where the function is decreasing will be illustrated in a similar manner.

In the final lesson of the unit, student will apply what they have learned to solve optimization problems. The problems will initially be presented as illustrations. Shrinking and growing will enable the class to understand how the surface area of a container such as a can might change while the volume remains the same. The first problems the students solve will be presented as pictures with verbal descriptions. Most problems given on the IB exam are presented with words and an accompanying sketch. It is my goal that the students will be able to relate the simple sketch on their exam to the more detailed pictures we worked with in class, increasing their understanding of what is being asked.

Classroom Activities

Lesson 1: Section 12.1 Limits and derivatives

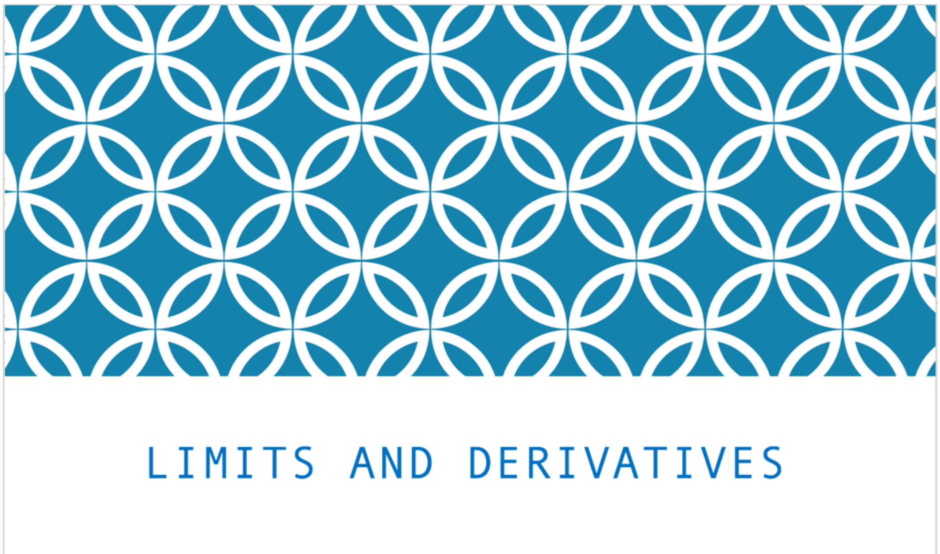
Learning Objective: At the end of this lesson, students will be able to find the derivative of power functions and functions comprised of sums of power functions. They will use the derivative to calculate the slope of the tangent line and they will identify intervals of the independent variable where the function is increasing, decreasing, or stationary. They will apply these skills to solve a variety of problems based on situations such as business applications or geometric figures.

Time: two class periods.

Materials:

- Mathematics: Applications and Interpretations Textbook (Oxford IB Diploma Program), section 12.1

- Graphing Calculator or Computer for each student
- PowerPoint slides, section 12.1, Limits and Derivatives



Procedures:

1. Do now – review Skills Check problems on page 524 (assigned for previous night's homework)
2. Use PowerPoint slides 1-2 to review linear function vocabulary: gradient and rate of change. Note, US students are familiar with the term slope but slope is called gradient in IB textbooks and on the exams. I use both terms while teaching to reinforce that they are the same.
3. Divide students into pairs to complete Investigation 1 on page 525-526. This activity will lead them to find the slope of the tangent at a given point and to find the slope of secant lines graphed closer and closer to the given point.
4. Use PowerPoint slides 3-9 to present vocabulary they will be using throughout this chapter: chord, secant, average rate of change, instantaneous rate of change, gradient of the tangent, and derivative.
5. Direct the pairs of students to complete Investigation 2 on page 527-528. Students will find slopes of horizontal lines and then slopes of functions of the form $f(x) = ax^n$, with x being equal to 1, 2, and 3. They will look for patterns and make a conjecture about the formula for the derivative of a function of this form.
6. Reconvene as a class, share what they have discovered, and write a rule for the derivative of a function of the form $f(x) = ax^n$.
7. Use PowerPoint slides 10-14 to present a sample of problems similar to those in example 1. These slides present the problems step-by-step and include graphs for some

questions to lead the class through the steps to answer the problems. Use the graphs to review the definitions of increasing, decreasing, maximum, and minimum. Point out on the graphs the value of the derivative at points of local extrema.

8. Assign problems from Exercise 12A to complete in class as a formative assessment. Assign the remaining problems for homework. Review the answers and check that the class understands how to find a derivative, gradient of a tangent, and intervals where the function is increasing.
9. Use PowerPoint slide 15 which illustrates problem 2 of example 2 (a mosquito flying towards a hammock). The illustration will guide the students to see how the shape of a hammock follows the curve of a quadratic function and the path of the mosquito is along a line tangent to the hammock. Review the problem, which is worked step-by-step in the textbook, with the class.
10. Arrange students into groups of three. Assign the groups to review and discuss problem 1 of example 2 (business production costs) and then to begin the problems in Exercise 12B, page 531. Homework will be to complete the problems.

Lesson 2: Section 12.2 Equations of Tangent and Normal

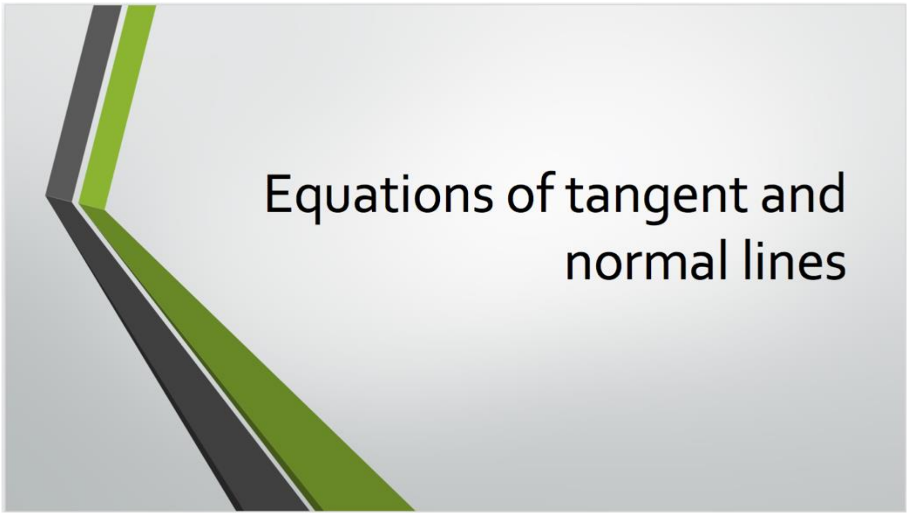
Learning Objective: At the end of this lesson, students will be able to evaluate the derivative of a function for a given value of x or at a given point. They will equate the derivative to the slope of the tangent at that point and find the equation of the line tangent to a given function through a given point. They will use the definition of normal line as the line perpendicular to the tangent line to find the equation of the normal line of a given function at a given point.

Time: one class period

Materials:

- Mathematics: Applications and Interpretations Textbook (Oxford IB Diploma Program), section 12.2
- Graphing Calculator or Computer for each student

- PowerPoint slides, section 12.2, Equations of Tangent and Normal



Procedures:

1. Do now – review finding derivatives by applying the power rule. Review last night's homework.
2. Present PowerPoint slides 1-6 – finding the equation of a tangent line when given an equation of a function and a point.
3. Divide class into groups of 3-4 students to read and discuss Example 3 on p. 532 and complete Investigation 3 on page 533. Check in with groups to offer clarification and check for understanding.
4. Reassemble as a class to view the PowerPoint slides 7-12 to define the normal line and find the equation of a normal line.
5. Guide the class through examples 4 and 5 – finding the equation of the normal line and solving for unknown values.
6. Assign problems from Exercise 12C to complete in class as a formative assessment. Assign the remaining problems for homework. Review the answers to the problems completed in class. Check that the class understands how to find a derivative, gradient of tangent and normal lines, and their equations.

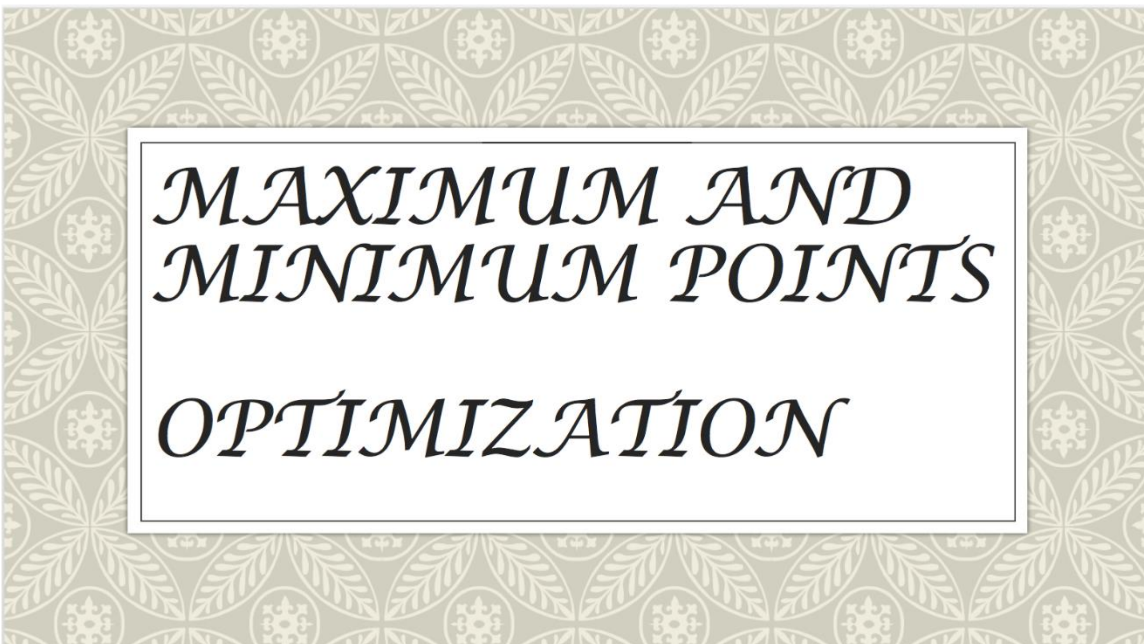
Lesson 3: Section 12.3 Maximum and minimum points and optimization

Learning Objective: At the end of this lesson, students will be able to identify and find the coordinates of local maximum and minimum points. They will use the concepts of maximum and minimum to solve problems involving optimization.

Time: one class period

Materials:

- Mathematics: Applications and Interpretations Textbook (Oxford IB Diploma Program), section 12.3
- Graphing Calculator or Computer for each student
- Graph Paper, rulers, colored pencils
- PowerPoint slides, section 12.3, Maximum and minimum points and optimization



Procedures:

1. Do now – Complete Investigation 4 on page 536. Review the answers. Review last night's homework.
2. Present PowerPoint slides 1-9 – Vocabulary – stationary points, local and global minimum and maximum points; how to find stationary points and identify them as maximum or minimum points.
3. Divide class into groups of 3-4 students to read and discuss Examples 7 and 8 on p. 537-538 and complete 2 problems from Exercise 12E. Assign the remaining problems for homework.
4. Reassemble as a class to view the PowerPoint slides 10-15 to define Optimization and view example problems.
5. Guide the class through example 9 – minimizing surface area.
6. Complete Investigation 5 on page 541 together as a class. Offer tips on using their GDC to graph a function along with its derivative and to find the coordinates of maximum and minimum points.

7. Recall geometric formulas needed to complete Exercise 12F. Assign Exercise 12F for homework.

Resources

Bibliography

Animations in SMART Notebook. (October 27, 2011). Retrieved February 25, 2020, from https://www.youtube.com/watch?v=PQwaFyzhnwQ

The above video presents several examples for animating pictures in SMART Notebook.

Haciomeroglu, E. S., & Chicken, E. (2012). Visual thinking and gender differences in high school calculus. *International Journal of Mathematical Education in Science and Technology*, *43*(3), 303. doi:10.1080/0020739X.2011.618550

This article discusses statistics from Advanced Placement Calculus students concerning visual preferences as it relates to gender.

Syllabus Outline. (n.d.). Retrieved May 1, 2020, from https://resources.ibo.org/dp/subject/Mathematics-2021-applications/works/dp_11162-53993?root=1.6.2.4.5&lang=en

Detailed syllabus of the IB course, mathematics: applications and interpretation

Syllabus Outline. (n.d.). Retrieved March 3, 2020, from https://resources.ibo.org/data/d_5_matsd_gui-out_1203_1_e.pdf

Detailed syllabus of the former IB course, math studies.

The Magic Shape reveal in SMART Notebook 10. (April 27, 2009). Retrieved February 25, 2020, from https://www.youtube.com/watch?v=6zrs5SgjNm0

Presents several examples of using animation and layering colors for SmartNotebook.

Visualizing Elementary Calculus: Graphs, Tangents, Derivatives. (2011, April 17). Retrieved March 2, 2020, from https://arcsecond.wordpress.com/2011/04/17/visualizing-elementary-calculus-graphs-tangents-derivatives/

Material for classroom use:

Belcher, P., Wathall, J. C., Doering, S., Duxbury, P., Economopoulos, P., Forrest, J., . . . Tokman, P. W. (2019). *Mathematics: Applications and interpretation: Standard level: Course companion*. Oxford: Oxford University Press.

This text is published by Oxford University in cooperation with the International Baccalaureate Organization. It is the primary textbook I use in my class and the one on which this project is based.

Hease, M., Humphries, M., Sangwin, C. J., & Vo, N. (2019). *Mathematics: applications and interpretation Sl*. Marleston: Hease Mathematics.

This excellent textbook may be used as the primary textbook for your class or as an additional resource for presenting material and finding additional practice questions.

Reading list for students:

Belcher, P., Wathall, J. C., Doering, S., Duxbury, P., Economopoulos, P., Forrest, J., . . . Tokman, P. W. (2019). *Mathematics: Applications and interpretation: Standard level: Course companion*. Oxford: Oxford University Press.

The textbook that the included lessons are based upon.

Appendix

PA Core Standards Mathematics, Pennsylvania Department of Education

CC.2.2.HS.D.1 Interpret the structure of expressions to represent a quantity in terms of its context.

CC.2.2.HS.D.4 Understand the relationship between zeros and factors of polynomials to make generalizations about functions and their graphs.

CC.2.2.HS.D.8 Apply inverse operations to solve equations or formulas for a given variable.

CC.2.2.HS.D.10 Represent, solve, and interpret equations/inequalities and systems of equations/inequalities algebraically and graphically.

CC.2.2.HS.C.1 Use the concept and notation of functions to interpret and apply them in terms of their context.

CC.2.2.HS.C.2 Graph and analyze functions and use their properties to make connections between the different representations.

NCTM Standards

analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior

understand the meaning of equivalent forms of expressions, equations, inequalities, and relations

write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency—mentally or with paper and pencil in simple cases and using technology in all cases

approximate and interpret rates of change from graphical and numerical data.