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A Visual Approach to Learning Mathematics
Conquering Space: A Voyage to Understanding

Rationale

High schools have changed over the last century and with them, the teaching of mathematics. Since the beginning of the 20th century, there have been debates over content and pedagogy. The debate over teaching Algebra and Geometry to all students or only to those who are college bound has seesawed with the decades and in some circles, is still happening now.

The progressivist's camp believes that students should learn what they want and that their math courses should have direct practical applications, like making a budget or completing a tax return. An example of that would be shop math or consumer math. In fact, prominent education experts in the early 1900's, Dr. William Heard Kilpatrick and Dr. Edward L. Thorndike argued that the study of algebra and geometry were of little value to boys and even less value, only 1%, to girls. Others, comprised of professionals and educators grounded in mathematics, see the value in the mental discipline required in Algebra and Geometry and the need for higher level math courses for everyone, for its applications and intrinsic value. (Klein, 2003)

In the 1980's the National Council of Teachers of Mathematics (NCTM) published two documents with the power to reform the teaching of Mathematics. The release of An Agenda for Action and the Curriculum and Evaluation Standards for School Mathematics changed the way mathematics was taught. Communication, making connections and creative, critical thinking were emphasized and rote was de-emphasized. (NCTM 1980,1989) The ground was laid for constructivism in mathematics classrooms! These documents were the forerunners of today's Common Core State Standards.

The goal in our TIP course, "A Visual Approach to Learning Mathematics", taught by Professor Robert Ghrist, was to improve our ability to motivate and communicate mathematical ideas and principles through the use of creative design, technical illustration, and animation. Professor Ghrist, who has his own YouTube channel, taught us basic skills in PowerPoint to assist us in creating lessons to help our students visualize mathematics, a key to many of the 8 Standards of Mathematical Practice published as part of the Common Core State Standards for Mathematics in 2010. Noted author, Stanford professor and You Cubed founder, Jo Boaler agrees with the importance of visualizing in mathematics. In her book *Mathematical Mindsets Unleashing Students' Potential Through Creative Math, Inspiring Messages and Innovative Teaching*, she states "When we don't ask students to think visually, we miss an incredible opportunity to increase their understanding. (Boaler, 2016, p. 63) In his Edutopia article, Jeremiah Ruesch, reminds us that "our jobs as educators is to set a stage that maximizes the amount of learning done by students, and teaching students mathematics in [a] visual way provides a powerful pathway for us to do our job well." (Reusch, 2017) The purpose of this curriculum unit is to do just that for volume and surface area in Geometry, using College Preparatory Mathematics' curriculum from CPM 2.

In 1999, the US Department of Education recommended 10 recommended mathematics programs. College Preparatory Mathematics (CPM) was one of the six exemplary programs listed. The books were student centered, the students worked in teams, the problems required discussion, and the homework was spiraled.

I use the CPM Geometry textbook and will base the lesson that follow on their unit entitled Spatial Visualization. As a teacher with many years of experience, it was refreshing to see that geometry and creativity abide together.

According to the 2015 TIMMS report, in Advanced Mathematics, the U.S. average percent correct in geometry was the lowest category, with 38% correct, compared to the Advanced Mathematics overall average of 44% correct.(nces.ed.gov, 2020) In NCTM's 2000 publication, "Principles and Standards for School Mathematics, the authors state: "Students' skills in visualizing and reasoning about spatial relation- ships are fundamental in geometry. Some students may have difficulty finding the surface area of three-dimensional shapes using two-dimensional representations because they cannot visualize the unseen faces of the shapes. Experience with models of three-dimensional shapes and their two-dimensional "nets" is useful in such visualization. Students also need to examine, build, compose, and decompose complex two- and three-dimensional objects, which they can do with a variety of media, including paper-and-pencil sketches, geometric models, and dynamic geometry software." (NCTM, 2000, p. 237)

Pierre van Hiele and his wife Dina developed a theory about difficulties students have in learning geometry. They named 5 levels of thinking in geometry and reasoned that a student could not reach subsequent levels without first understanding the basic level of visualization (Zalman, 1981). This ties in precisely with how my curriculum unit begins.

The confusion between surface area and volume is an especially prevalent issue in Geometry. I have found this to be true in my own classes and researchers have verified this issue. Allison Dorko and Natasha Speer, at the time, both from the University of Maine found that Calculus students often thought that by adding the area of the faces of a solid, they were incorporating a 3^{rd} dimension. (Dorko & Speer, 2013) In later research, they noted the significance that units of measure play in student understanding. 73% of the almost 200 Calculus students in their study made mistakes in labeling the units in the assigned area and volume tasks. (Dorko & Speer, 2015) This curriculum unit uses spatial visualization and manipulatives to introduce surface area and volume in order to strengthen student understanding. Barbara Kinach, an associate professor of mathematics education, states in her article in the Mathematics teacher "More emphasis on spatial reasoning is a way to increase meaning when students study Geometry." (Kinach, 2012, p. 534)

In the PSAT and SAT, this topic in Geometry is in the category Problem Solving and Data Analysis. The students at the Philadelphia High School for Girls have not been scoring well in this category. As shown in the graphs below, the percentage of students performing well in this category, did not increase, even after taking Geometry.

PSAT Data Juniors 2019

Problem Solving and Data Analysis — Mean Score 8

	Need To Strengthen Skills		Close to Being on Track for College Readiness		On Track for College Readiness	
School	42%	77 Test Taker(s)	57%	104 Test Taker(s)	2%	3 Test Taker(s)
District	60%		36%		4%	
State	31%		57%		12%	
Total Group	33%		55%		12%	

PSAT Data Sophomores 2018

Problem Solving and Data Analysis — Mean Score 7

	Need To Strengthen Skills		Close to Being on Track for College Readiness		On Track for College Readiness	
School	32%	63 Test Taker(s)	66%	128 Test Taker(s)	2%	4 Test Taker(s)
District	54%		38%		7%	
State	26%		54%		20%	
Total Group	34%		51%		15%	

Content Objectives

The purpose of my curriculum unit is to improve the student understanding of volume and surface area through visualization. I am hoping that the ability to construct and deconstruct a solid, fill it with cubes and measure the area of its faces, will strengthen a student's concept of the topic and therefore improve student performance in Problem Solving and Data Analysis tasks.

I already use various websites to visualize mathematics in my classroom, such as Desmos, Which one Doesn't Belong, Would you Rather, and Estimation 180. My students love the opportunity to have fun and do something they think is not math. I also use models in my classroom: cubes, algebra tiles, tissue boxes, etc., whatever I can find. That doesn't work so well when learning is all virtual. I was especially excited to create PowerPoint slides to show how 2 dimensions can become 3 dimensions and to analyze 3-D objects. Because one problem can take as many as 10 slides, each problem is its own PowerPoint. They are not all perfect, but I believe they get the idea across. Professor Ghrist told me not to let the perfect become the enemy of the good and I have followed his advice.

The unit begins by asking visual puzzlers, two- and three-dimensional geometric descriptions of situations and 3-D interpretations of 2-D sketches, including nets. The purpose of the first lesson is to develop, acknowledge and appreciate the value of spatial visualization with fun activities. One of the Do Now activities asks students to discuss ways children develop spatial and visualization skills and how often boys have different experiences than girls. With handheld video games as much a part of our current high school's generation as TV was for me, one would think that the gender gap in spatial thinking would be reduced. In quite technical, neuroscientific detail, Li Yuan and his research colleagues at Shaanxi Normal University in Xi'an China, found that males out-perform females in large- and small-scale spatial abilities. (Yuan, 2019)

Polyhedra are introduced with a net and students are asked to build them from gum drops and toothpicks. Vocabulary "vertices" "edges" and "faces" are introduced, and students are asked to discover Euler's formula. Polyhedra are named by the number of faces and bases. Students also use isometric grid paper to draw 3-D solids. Volume is introduced as the number of cubes in a solid. Mat plans are also utilized to record a 3-D solid. Surface area is introduced with the nets. After the surface area is found, the students build the polyhedron. Students will also find the slant-height of a pyramid. The unit culminates with a theme problem: finding the surface area of the Transamerica Pyramid in San Francisco, although I altered mine to use a local Philadelphia skyscraper.

Teaching Strategies

This curriculum unit uses many strategies for visualization and cooperative learning. Many of these strategies I have learned from CPM training and others are specific to the exact topic.

When students work regularly in study teams, there are a few basic guidelines that CPM suggests teachers establish in their classrooms: Each member of the team is responsible for his/her own behavior, Each member of the team must be willing to help any other team member who asks for help, You should only ask the teacher for help when all team members have the same question, Use your team voice. Other CPM team strategies, which Jo Boaler also mentions in Mathematical Mindsets are to have students think about a problem by themselves and then share with their teams. CPM also recommends reserves the first few minutes a student and the team spends with a problem be without pencils in the "Teammates Consult" strategy. In this way, the team can clarify the problem and discuss possible solution methods without one student getting ahead of the others with their own approach.

In this curriculum unit students will be doing a number of hands on activities with extra materials most days. One of the roles a team member can serve is Resource Manager whose job is to get supplies for the team and make sure the team is cleaned up. The resource manager also is responsible for calling the teacher over for questions. You may want to switch team roles daily to vary student tasks. The other team roles are Recorder/Reporter, Facilitator, and Task Manager.

All of the 8 Standards of Mathematical Practice can be used during lessons in this unit. Students will make sense of problems and persevere. This is especially applicable in the surface area problems and the theme problem. Students will reason abstractly and quantitatively. The best example of this is SV-104 where students have to decide if a 5-foot stick can fit in a cube that is 3ft on each side. Students will construct viable arguments and critique the reasoning of others. When students are discussing their answers to the spatial visualization questions, these skills will definitely be put into practice. Students will model with mathematics using all of the hands-on activities. The theme problem is the "cherry on top" for this practice. Students will use appropriate tools strategically, especially while measuring the nets of their 3-D solids to find the surface area. Students will also attend to precision with this activity. Students are asked to look for and make use of structure when generalizing a rule for the slant height or the length of a 3-D diagonal. Students will look for and express regularity in repeated reasoning when discovering Euler's Formula for the number of vertices, faces and edges of a cube in SV-79.

Other teaching strategies utilized in the unit are cooperative assessments, jigsaw, using academic vocabulary, and applying skills to a real-world problem. Of course, the most consistent and important strategy used in this unit is using visualization; a most important skill for the growing world of online learning.

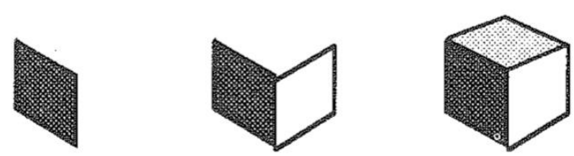
Classroom Activities

Lesson 1: Spatial Visualization
Learning Objective: At the end of this lesson, students will be able to visualize and 2-dimensional and 3-dimensional geometric representations of given situations. They will be able to describe the geometric representations using academic vocabulary and/or labeled drawings. Students will also be able to interpret 2-dimensional drawings as representing real world objects.

Materials:
- CPM Geometry Unit 4, Days 3 & 4 SV-22, 25-34, 39-43 (appendix 1)
- PowerPoint slides to ask and answer questions (appendix 2)
- manipulatives such as string, a ruler, paper and scissors
- *optional* optical illusions (appendix 3)

Procedures:
1. Use SV-22 as an assignment due the day of this lesson.

SV-22. Describe what you see in each case. Try to see something in a three dimensional sense.

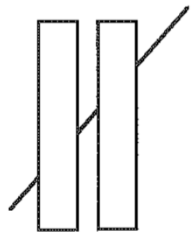


Case 1 Case 2 Case 3

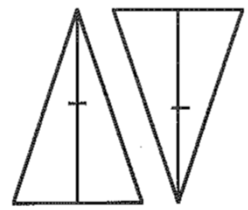
2. Use SV-25 & SV-39 as a Do Now so students don't have the physical pictures on their desks. These are questions that ask "yes" or "no" or "which line". Students should answer questions on their own, and then compare their answers with their teams. Do not expect agreement, but encourage justifications with their arguments.

SV-25. **DIRECTIONS:** Today you will be working on an important skill in mathematics: visualization. Many of the following problems do not ask you to <u>solve</u> anything, but rather to describe a situation as you imagine it or see it in your mind's eye. For each of the problems, include a sketch AND write descriptions in complete sentences. Examine each figure below and explain how you know that what you decide is really true.

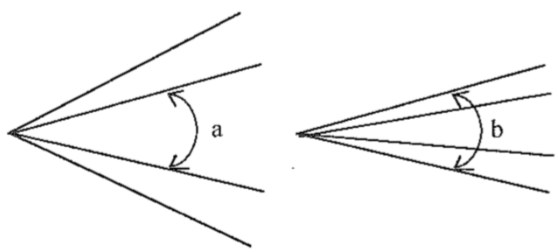
a) Is the diagonal line straight?



b) Are the horizontal slashes exactly in the middle of the center line of these triangles?

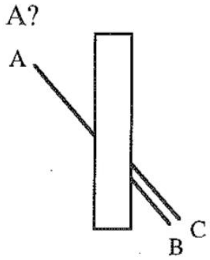


c) Which angle is larger, a or b?

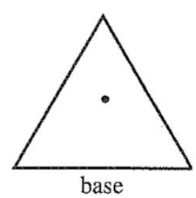


SV-39. Examine each figure below and explain how you know that what you decide is really true.

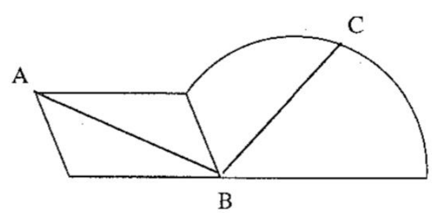
a) Which line is the continuation of A?



b) Is the point midway between the top vertex and the base?



c) Which line segment is longer, $\overline{AB}$ or $\overline{BC}$?



3. Use the prepared PowerPoint slides which reveal the answers with the geometric "proof". Teachers can take a thumb poll before going over each question to make sure each student has committed to an answer.

4. Assign SV-25-34 and SV-40-44 to students to work on as teams. Suggestions for problem assignments are in the notes below. The PowerPoint slides provide the problems and explanations of the answers.

Notes: Lesson 1 can take 1 or 2 class periods.

- This unit is a wonderful time to bring up a few classic optical illusions. I was surprised how many of my students hadn't see Young Woman vs. Old Woman (appendix 3). You could challenge students to find other optical illusions for a homework assignment or extra credit.
- If a teacher uses 2 class periods for Lesson 1, Assign the problems sequentially with SV-25 as the Do Now for day 1 and SV-39 as the Do Now for day 2.
- If a teacher wants to use only 1 class period, they can assign only specific problems to individual teams who will then be responsible for presenting those problems and the answers to the class.
- If a teacher wants to use only 1 class period, they can assign some of the problems for homework, choosing those that take less discussion, like SV-28,30,34, and 44.

Lesson 2 – Isometric Drawings

Learning Objective: At the end of this lesson, students will be able to draw a 3-D cube on 2-D isometric grid paper. Students will be able to copy and draw on isometric grid paper, a 3-D solid made up of stacks of cubes. Students will be able to use a MAT plan to create a drawing on isometric grid paper. Students will be able to find the volume of any 3-D solid given its MAT plan or its drawing on isometric grid paper.

Materials:

- CPM Geometry Unit 4, Days 5 & 6: SV-53-59, 65-72 (appendix 4)
- PowerPoint slides to ask and answer questions (appendix 5)
- isometric dot grid paper (appendix 6)
- cubes for students who will benefit from live model or manipulatives (1 cm cubes, multi-link cubes, or sugar cubes)
- 3x5 cards as base for cube structures in order to rotate for perspective
- large grid paper for creating Mat Plans (optional) (appendix 6.5)

Procedures:

1. Use SV-53 as a Do Now question. Make sure teams discuss their answers and then lead the class in one discussion. You may need to clarify misconceptions about gender differences.

SV-53. **DIRECTIONS**: Much of the focus of this course is two-dimensional geometry (figures on flat surfaces), but reality comes in at least three dimensions. An important and useful skill is to be able to draw 2-dimensional representations of 3-dimensional objects, and to be able to interpret 2-dimensional drawings as representing real-world objects. Research indicates that men tend to be better at doing this type of visual translation. These studies point out that in general girls and boys have different experiences as children and that boys have many more opportunities to do the type of visualization that later helps them in mathematics. Is this generalization true for the people in your team? Remember findings that are true for the general population do not necessarily apply to individuals.

a) Make a list of experiences you have had that might have helped you develop visualization skills. Perhaps they include the games, toys, and/or sports you played as a child.

b) Research also shows that people who have had relatively little experience with this kind of visualization catch up very quickly when they have the opportunities to practice. The following problems provide many such opportunities. Go on to the next problem now.

2. Distribute Isometric Dot Paper Grid and guide students through the directions and drawing their first 3-D isometric cube. There are two methods, the hexagon method shown in SV-54 and making one face at a time. Encourage students to try both methods to see which seems more natural to them. Students should draw as many single cubes as they need to feel comfortable drawing them. Remind them to shade the top of each single cube.
3. Distribute cubes to teams and instruct teams to draw the cube stacks in SV-54. Encourage students to build the stacks with the cubes on a 3x5 card to allow turning. The standard view is top, right and front. You can mark the four directions (front, right, back and left) on the 3x5 card. Ensure that students drawings are accurate at this point – allow teammates to help one another, but make sure students do their own drawing.
4. Allow students to work on SV-55-59 on their own, using their teammates as checkers and helpers. Remind students to shade the top of each cube stack.
5. Introduce a Mat Plan with SV-65. Note that the number of cubes in each stack is recorded in its square grid. Compare the Mat Plan with the isometric view. Explain and discuss the idea of a hidden cube. The slide shows the rotation which reveals where a cube could be hidden.
6. Review the directions in SV-66 showing the progression from Mat Plan to isometric view with a Mat Plan on the isometric dot grid paper which rotated 45 degrees clockwise. Students will practice with a) and b)
7. SV-67 introduces **volume** as the number of 1x1x1 cubes in a solid. Volume asks the question "how many cubes?" which stresses the answer be in cubes.
8. Students should work on SV-68-73 on their own in their teams, again checking and helping each other. SV-71 introduces a definition for **Prism** in terms of a Mat Plan.

Notes:

- The CPM 2 book was published back in 1999, but current research still verifies that males outperform females in overall spatial ability (Yuan, et al, 2019). Making a list of influential experiences is a worthy task.
- When drawing the cube stacks, some students may use a top, left and front view. If their drawings are accurate, this is not a problem. The focus is visualization!

- Drawing solids from a Mat Plan onto isometric dot grid paper is difficult for many students. Mastery is not necessary; the main ideas from that portion of the lesson are volume and prism.
- CPM suggests this lesson take 2 days. If you are short of time, you could skip the Mat Plan exercises, using the isometric drawings from SV-54-59 to introduce Volume and Prism. You will need to ensure a sketch of a prism is included; SV-58 provides an opportunity to do so.

Lesson 3 – Building Polyhedra

Learning Objectives: At the end of this lesson, students will be able to use polyhedral vocabulary to accurately find the number of vertices, edges and faces of a given polyhedron. Students will discover and use Euler's Polyhedral Formula to check the accuracy of a polyhedron's number of faces, vertices and edges. Students will also be able to identify and name polyhedral by their number of sides or base.

Materials:
- CPM Unit 4, Day 7, SV-79-84 (appendix 7)
- PowerPoint slides to introduce the activity and the vocabulary (appendix 8)
- Team Activity Quiz (appendix 9)
- SV-79 Resource pages (appendices 10 & 11)
- spice drops (40/team or fewer if reusing) } package or sort ahead of time.
- rounded toothpicks (46/team or fewer if reusing)
- colored paper for copies of SV-79 Tetrahedron (optional)
- scissors
- tape or glue sticks or glue

Procedures:
1. Introduce students to polyhedral vocabulary: vertex, edge and plane. SV-79 does this.
2. Distribute the tetrahedron net (SV-79) to teams to build, either one template per pair or per team. Instruct the students to fold on the dotted lines and to use the tabs (warn students not to cut them off) to create the 3-D solid. Teams will also need scissors and adhesive.
3. Demonstrate and count the vertices, edges and faces of the tetrahedron.
4. Introduce the Building Polyhedra activity by explaining the materials they will use and what part of the polyhedra each represents (gum drops = vertices, toothpicks = edges).
5. Distribute and introduce the rubric and materials for the activity quiz. Each team will need 1 rubric, 1-2 copies of the SV-79 chart, at least 40 gum drops and at least 46 toothpicks. Each team is responsible for building 4 of the polyhedra and then, together, counting and recording the number of vertices, edges and faces for each polyhedra. If a team builds 4, they will have to use the picture to count for the remaining 2 polyhedra. The students should also add the number of vertices and faces for each polyhedra to record in the column V + F
6. After the students have completed the SV-79 chart, ask them to find the relationship between the number of vertices, faces and edges. The column V + F is key to the

relationship. Once they "discover" Euler's Polyhedral Formula, they can use it to check their data in the chart.

7. The last problem for the activity quiz is finding the number of vertices, faces and edges of the figure in the sketch SV-84.
8. Students should complete the remaining problems SV-81, 83 and 84 about polyhedra and the names according to their faces. You can help them make the connection to the polyhedra names from their chart and their number of faces.

Notes:

- CPM SV-79 has a different order of introduction for the tetrahedron template and model. I like using the template first so I can physically show the faces, which are "imagined" for the models.
- I like to use paper of plastic cups for distributing the gum drops and toothpicks.
- You may care to have newspaper on the desks before the model building. There will be quite a bit of sugar left over.
- Emphasize that the students should not eat the gum drops. They are not individually packaged and were touched before distribution. I sometimes buy actual candy for the students to consume so they aren't tempted by the vertices. Spice drops are generally not too appealing to high schoolers.
- SV-81, 83, and 84 could be assigned for homework if students don't have time in class to complete.
- The polyhedra, once dried, are fun to hang around your classroom using fishing wire.

Lesson 4 – Building Polyhedra, Volume

Learning Objectives: At the end of this lesson, students will be able to list similarities and difference between pyramids and prisms. Students will also be able identify the base of any polyhedra and name them accordingly. Students will continue to find the volume of prisms, for this lesson using the formula Volume = (Area of Base)(Height of the Prism). Students will find and accurately draw pyramids and prisms using dotted segments for hidden edges and open circles for hidden vertices.

Materials:

- CPM Unit 4, Day 8 (SV-91- 97 + 100) (appendix 12)
- SV-91,92,93 Resource pages (appendix 13)
- SV-94 pyramid Resource page (appendix 14)
- SV-94 prism Resource page (appendix 15)
- Graph paper for SV-97 if assigned
- colored paper for copies of SV-94 pyramid and SV-94 prism (optional)
- scissors
- tape or glue sticks or glue

Procedures:

1. Use SV-94 as a Do Now. Distribute the pyramid and prism template to a pair of students. Each student will build only one.

2. Have students compare their polyhedra and make comparisons (individually, as a pair, or as a team), answering parts a and b. You can use a Venn Diagram to chart the comparisons. Have teams answer part c
3. Distribute resource page for SV-91,92,93. Students will work individually to draw hidden pyramids and hidden prisms, but they may need help seeing the base from teammates.
4. SV-95 and 96 help students to understand that the base isn't always the bottom of the polyhedron. While identifying the base(s) of a given polyhedron, a student is also taking the first step at naming the polyhedron.
5. Finish class with SV-100, finding the volume of two prisms. SV-94 introduces the formula $Volume_{prism} = (Area\ of\ Base_{prism})(Height_{prism})$. Do SV-94a, a triangular prism with the class and allow the teams to find the volume of an oblique parallelogram-based prism for SV-94b. Introduce the term oblique and answer any questions the teams may have.
6. Give another prism volume question for an exit ticket for students to solve individually. You may want to use a different polygon for a base, like a hexagon.

Notes:

- SV-94 was not meant to be done before SV-91-93, but I like having the teams discover the properties of prisms and pyramids before looking for the hidden ones.
- If you are worried about students taking too long to make the sketches of the bases for SV-96, you could give them a copy of the problem and instruct them to shade a base for each polyhedron.
- Notice in the formula for volume in SV-94, used for SV-100, the b is capitalized for the base of the prism, not the triangle. The same is true with the capital h for height of the prism, not the triangle. When reading the formula, I either include the word prism to describe the Base and Height or say, "capital b base." You can also use altitude for one the heights to avoid confusion.
- I prefer the generic formula for volume rather than the formula given most often for only a rectangular or square-based prism because it helps students realize the role the Base plays and is much more useful when finding volume.
- SV-100b allows opportunity to distinguish between a rectangle base and a parallelogram base even though their area formulas are the same. The fact that the prism is oblique gives further discussion to using the perpendicular measure for both height and Height.
- I've included SV-97, which is a homework problem. This asks students to draw a 2-D version of a triangular prism and a square based pyramid. This could be extended beyond naming by analyzing the bases of the polyhedra and discussing how to find the areas.

Lesson 5 – Surface Area

Learning Objectives: At the end of this lesson, students will be able find the surface area of a polyhedron given its net after first measuring the appropriate dimensions. Students will use the hands-on experience to find the surface area of any given pyramid or prism drawing by first analyzing the faces of the polyhedron, then finding the area of each face, and finally, finding the sum of the areas.

Materials:

- CPM Unit 4, day 9 (SV101, 103) (appendix 16)
- PowerPoint slides to accompany the lesson (appendix 17)
- SV-101 solid templates (appendix 18)
- SV-101 chart (appendix 19)
- Colored paper for SV-101 solid templates (optional)
- Metric rulers
- Scissors
- tape or glue sticks or glue

Procedures:

1. Assign problems for finding the area of polygons as a Do Now. Include triangles with various heights/altitudes. Stress the need for appropriate units in their answers. This is a warmup for finding areas of faces for surface area. See notes for an alternative Do Now.
2. Distribute one of each of the four templates for the SV-101 solids (triangular pyramid, octahedron, rectangular prism, and square pyramid) to the teams. Each student should be responsible for a different solid.
3. Display the PowerPoint slides to introduce the SV-101 task. Review how to measure and what segments the students will be measuring. The heights of each triangle are labeled x, but it is not the always the same value.
4. Students will measure the appropriate sides of their net in millimeters and find and record the area of each face. (SV-101a,b)
5. Students will find total surface area by finding the sum of all of the faces of their solid. The strategy is to group together like faces. (SV-101d)
6. Individual students contribute their polyhedron's data for the chart and explain to their teammates how they did the problem. (SV-101e)
7. Students should build their polyhedron. (SV-101c)
8. Assign teams SV-103. Allow the team to figure out the 2-D parts of the solid. The slides will show various ways to consider the parts, including a net, but don't show these until after the teams have attempted/completed the problems.
9. Give one more surface area problem as an Exit Ticket.

Notes:

- Depending on the measuring skills of your students, an alternative Do Now would be to have them practice measuring in millimeters. The polygons could be on paper without dimensions.
- I use colored paper to keep the various templates organized.
- You could jigsaw the activity by having students work in groups according to their solid and then go back and share their data with their original team.
- I usually have the students cut out their templates first so they can see the actual faces if they need to. I have also found that students will write their work on the space around the net and then throw that away before recording their data in the table.

Lesson 6 – Three-dimensional diagonal and Theme Problem.

Lesson Objectives: At the end of this lesson, students will be able to find the slant heights of the lateral faces of a pyramid and a 3-dimensional diagonal for a rectangular prism. Students will also be able to solve a real-world problem utilizing surface area.

Materials:

- CPM Unit 4, Day 9 (SV-102, 104, 105) (appendix 20)
- PowerPoint slides to accompany lesson (appendix 21)
- SV-102 adapted for Philadelphia (appendix 22)
- Possible physical models for SV-104: A box with a stick that fits along its diagonal, an open pyramid to show pyramid Height and slant height.

Procedures:

1. We began this unit with spatial visualization questions. For the last day of the unit, we will begin the same way. Have students guess yes or no to SV-104 “Will a 5 ft stick fit in a 3’x3’x3’cube?” A physical model may be helpful; it doesn’t have to match the dimensions of the problem. Discussion in their teams should be encouraged.
2. Either with an explanation from a student or using the slides, show the class how it could fit (along a 3-dimensional diagonal).
3. Reassign the problem to the teams to actually do the math. Use the slides or a physical model to show the view. Review the answer with the class.
4. Revisit SV-103a and ask the teams to see what the two problems have in common.
5. Introduce slant height to describe the 8cm length in SV-103a and ask the students to use the given information to find the Height of the pyramid. You can use the square pyramid from SV-101 to model the problem.
6. Introduce the theme problem SV-102. You can make it a classwork exercise or a team quiz.
7. If there is time, give SV-105 as an exit ticket or part of the team quiz. Award extra credit if a team can generalize a formula.

Notes:

- The more about the 3-D diagonal that can be discovered, the better. Visualization definitely improves understanding in this case.
- If you are using the square pyramid from SV-101 as a physical model, have one of the lateral faces open and the other 3 connected to the top vertex. You could even make a paper right triangle to fit in it to show where the slant height comes from.
- The original theme problem uses the Transamerica pyramid building in San Francisco. I find value in using a more local skyscraper. Try to find a building which has a pyramid somewhere in its design. You can also add cost to the problem to make it even richer.
- Because I used a local building, my Do Now was to ask students to identify buildings in the Philadelphia skyline. I was surprised by how many of my students were not familiar with our city hall!

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Additional Resources

Bouck, Emily, Sara Flanagan, and Mary Bouck. "Learning Area and Perimeter with Virtual Manipulatives." The Journal of computers in mathematics and science teaching. 34.4 (2015): n. pag. Print.

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A website with many interactive math lessons, including surface area and volume.
http://www.shodor.org/interactivate/lessons/SurfaceAreaAndVolume/

Academic Standards

From the Common Core State Standards for Mathematics:

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs.

Geometric measurement and dimension G-GMD

Explain volume formulas and use them to solve problems

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

2. (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

Visualize relationships between two-dimensional and three- dimensional objects

4. Identify the shapes of two-dimensional cross-sections of three- dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

modeling with Geometry G-MG

Apply geometric concepts in modeling situations

4. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

5. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

6. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

From PA Common Core Standards:

CC.2.3.HS.A.13
Analyze relationships between two-dimensional and three-dimensional objects.
CC.2.3.HS.A.3 Verify and apply geometric theorems as they relate
to geometric figures.
CC.2.3.8.A.1
Apply the concepts of volume of cylinders, cones, and spheres to solve real-world and mathematical problems.
CC.2.3.HS.A.12
Explain volume formulas and use them to solve problems.
CC.2.3.HS.A.14 Apply geometric concepts to model and solve real- world problems.

From PA Geometry Assessment Anchors and Eligible Content

G.1.2.1.1 Identify and/or use properties of triangles
G.1.2.1.2 Identify and/or use properties of quadrilaterals
G.1.2.1.3 Identify and/or use properties of isosceles and equilateral triangles
G.1.2.1.4 Identify and/or use properties of regular polygons
G.1.2.1.5 Identify and/or use properties of pyramids and prisms
G.2.2.3.1 Describe how a change in the linear dimension of a figure affects its perimeter, circumference, and area (e.g. How does changing the length of the radius of a circle affect the circumference of the circle?).
G.2.3.1.1 Calculate the surface area of prisms, cylinders, cones, pyramids, and/or spheres. Formulas are provided on a reference sheet.

G.2.3.1.2 Calculate the volume of prisms, cylinders, cones, pyramids, and/or spheres. Formulas are provided on a reference sheet.
G.2.3.1.3 Find the measurement of a missing length given the surface area or volume.
G.2.3.2.1 Describe how a change in the linear dimension of a figure affects its surface area or volume (e.g. How does changing the length of the circumference of the edge of a cube affect the volume of the cube?).